PHYSICS 282 Spatiotemporal Biodynamics

Homework #1

Due Wednesday Oct 16, 2024

[Note: Those not from math/physics background need not attempt problem(s) indicated by *]

1. Phase diagram and phase transitions in dynamical systems. In class, we studied a model of growth and predation. Let the density of the organism we study be $\rho(t)$. If the growth of the organism is described by a logistic term and the effect of predation by a hyperbolic term, then the dynamics is given by the following ODE,

$$\frac{d\rho}{dt} = r\rho \cdot \left(1 - \frac{\rho}{\tilde{\rho}}\right) - \frac{\delta \cdot \rho}{1 + \rho/\rho_\delta},$$

where r is the maximal replication rate and δ is the maximal predation rate. Of the two remaining parameters, $\tilde{\rho}$ describes the carrying capacity and ρ_δ describes the saturating density for predation.

Upon introducing dimensionless variables, $u \equiv \rho/\tilde{\rho}$, $\tau = rt$, the above equation becomes

$$\frac{du}{d\tau} = u \cdot (1 - u) - \frac{\alpha u}{1 + u/\kappa},$$

with two dimensionless parameters $\alpha \equiv \delta/r$ and $\kappa \equiv \rho_\delta/\tilde{\rho}$.

(a) Write a general expression for the fixed point $u^*(\kappa, \alpha)$ at which $\frac{d}{d\tau}u = 0$, and show that the nature of the solution u^* depends importantly on whether the relative predation rate α is smaller or larger than

$$\alpha_c(\kappa) = \frac{1}{2} + \frac{\kappa}{4} + \frac{1}{4\kappa}.$$

Show $\alpha_c(\kappa)$ has a single minimum at $\kappa = 1$. The point $(\kappa = 1, \alpha = \alpha_C(\kappa = 1))$ is called the "critical point" due to the special behavior exhibited by the system in the vicinity of this point as will be worked out below.

(b) For $\kappa = 1$, find $u^*(\alpha)$ vs α. [Hint: There are two non-negative real values for $u^*(\alpha)$ for $\alpha < \alpha_c$, and only one for $\alpha > \alpha_C$. Ignore any complex solutions which are not biological.] To see which of the fixed points is stable/unstable, plot $\frac{d}{d\tau}u$ vs u for $\kappa = 1$ and the following values of α: (i) $\alpha \lesssim \alpha_c(1)$, (ii) $\alpha = \alpha_c(1)$, (iii) $\alpha \gtrsim \alpha_c(1)$. For each case, plot the "flow", i.e., the direction of $\frac{d}{d\tau}u$, as arrows for different regions of u. The type of *phase transition* which occurs at the critical point here is called a "supercritical bifurcation".

(c) Sketch (i.e., plot the approximate dependence by hand, not by computer) the dependence of the stable fixed point u^* in the vicinity of the *critical point* $\alpha_c(\kappa = 1)$. The robustness of the system can be characterized by the sensitivity of the density u^* to small changes in the environment. Let $S \equiv \frac{d}{d\alpha}u^*$ be a measure of the change in population density when the predation rate changes. Sketch $S(\alpha)$ in the vicinity (i.e., on both sides) of the critical point, and describe the behavior in words.

(d) For $\kappa = 1/2$, show that there are three solutions for $u^*(\alpha)$ for a range of α at $\alpha_0 \leq \alpha \leq \alpha_c(1/2)$, where α_0 is a positive number you need to determine. To see which of the solutions are stable/unstable, plot $du/d\tau$ vs u for $\kappa = 1/2$ and the following values of α : (i) $\alpha \lesssim \alpha_c(1/2)$, (ii) $\alpha = \alpha_c(1/2)$, (iii) $\alpha \gtrsim \alpha_c(1/2)$, (iv) $\alpha \lesssim \alpha_0$. (v) $\alpha = \alpha_0$. (vi) $\alpha \gtrsim \alpha_0$. For each case, plot the "flow" as in **(b)**. Indicate the stable and unstable fixed points which arise in each case. In cases where there are multiple stable fixed point for the same value α, what determines the value of u^*, the steady-state density?

(e) For the nonzero stable fixed point u^* obtained in **(d)**, use Taylor expansion to obtain the leading dependence on α in the vicinity of $\alpha_c(1/2)$ and in the vicinity of α_0. From these results, obtain the sensitivity $S(\alpha)$ of this fixed point and sketch both $u^*(\alpha)$ and $S(\alpha)$ in the vicinity of $\alpha_c(1/2)$ and in the vicinity of α_0. The type of *phase transition* which occurs at $\alpha_c(1/2)$ is called a "saddle-point bifurcation", while the phase transition which occurs at α_0 is called a "subcritical bifurcation". Describe in words how they are different from each other and from the supercritical bifurcation encountered in **(b)**.

(f) For $\kappa = 2$, show that there are two non-negative solutions $u^*(\alpha)$ for $\alpha < \alpha_1$, where α_1 is a number smaller than $\alpha_c(\kappa = 2)$, and one solution for $\alpha > \alpha_1$. Determine which of the fixed points is stable by plotting $\frac{d}{d\tau}u$ vs u for $\kappa = 2$ and the following values of α: (i) $\alpha \lesssim \alpha_1$, (ii) $\alpha = \alpha_1$, (iii) $\alpha \gtrsim \alpha_1$. For each case, plot the "flow" and indicate the stable and unstable fixed pointd as above. Using Taylor expansion, find the leading dependence of the stable fixed point u^* on α in the vicinity of α_1. From this result, find the sensitivity $S(\alpha)$, and sketch $u^*(\alpha)$, $S(\alpha)$ in the vicinity of α_1. The *phase transition* which occurs at α_1 is called a "transcritical bifurcation". Describe in words again how it is different from the bifurcations encountered in **(b)** and **(e)**.

(g) Based on the results obtained in **(a)**-**(f)** above, sketch the phase diagram in the parameter space (κ, α) as follows: Draw the line (actually a curve) $\alpha_c(\kappa)$ and put down the special points $(\kappa, \alpha) = (1, \alpha_c(1)), \left(\frac{1}{2}, \alpha_c\left(\frac{1}{2}\right)\right), \left(\frac{1}{2}, \alpha_0\right), (2, \alpha_1)$. With the additional knowledge that the critical values α_0 and α_1 are κ-independent (you don't need to derive this), you can obtain 2 lines that divide the entire parameter space into 3 distinct regions. Show the 2 lines in the space of (κ, α) and give a verbal description of the 3 "phases" separated by these lines. Indicate the nature of phase transitions (bifurcations) upon crossing each line separating the 3 phases.

2. Oscillatory genetic circuit. A genetic circuit in a cell involves two transcription factors, an "activator" and a "repressor". The activator activates the expression of itself and the repressor, while the repressor represses the expression of the activator. This circuit is known as the "predator-prey" circuit. Denoting the concentrations of the activator and repressor by $[A]$ and $[R]$, respectively, we can write down a simple set of equations describing their dynamics:

$$\frac{d[A]}{dt} = \alpha_A \frac{[A]}{[A] + K_A} \cdot \frac{K_R}{[R] + K_R} - \mu[A],$$
$$\frac{d[R]}{dt} = \alpha_R \frac{[A]}{[A] + K_A} - \mu[R].$$

In the above, the parameters K_A and K_R are the dissociation constants for the binding of the activator and the repressor to the promoter regions, respectively, α_A and α_R characterizes the activity of the two promoters, and μ is the rate of cell growth (which serves here to dilute the concentrations of the transcription factors.) In this problem, you will find conditions under which this circuit sustains oscillation.

(a) Make these equations dimensionless using $u = [A]/K_A$, $v = [R]/K_R$, and $\tau = \mu t$. Write down the dependences of the two remaining dimensionless parameters, $\sigma_A \propto \alpha_A$ and $\sigma_R \propto \alpha_R$, in terms of the original parameters of the problem.

(b) In (u, v) space, sketch the two null clines (i.e., the relation between u and v that makes $\frac{d}{d\tau}u = 0$ or $\frac{d}{d\tau}v = 0$. Indicate each sub-regions of (u, v) space partitioned by the two null clines whether $\frac{d}{d\tau}u$ and $\frac{d}{d\tau}v$ are positive or negative. Sketch qualitatively the "flow" of u and v by arrows in the (u, v) space.

(c) Find the fixed point(s) (u^*, v^*) where $\frac{d}{d\tau}u = 0$ and $\frac{d}{d\tau}v = 0$. Show conditions on the parameters σ_A and σ_R in order for there to be a "nontrivial" fixed point $u^* > 0$ and $v^* > 0$. Obtain how the nontrivial fixed point (u^*, v^*) depends on the parameters (σ_A, σ_R).

(d) In the vicinity of the nontrivial fixed point obtained in **(c)**, use Taylor expansion to linearize the dynamical equations for $x(t) = u(t) - u^*$, $y(t) = v(t) - v^*$. Find the two eigenvalues λ for the linearized system.

(e) Based on whether the eigenvalues λ found in **(d)** has nonzero imaginary component, and whether the real component of λ is positive or negative, find regions of the parameter space (σ_A, σ_R) where you expect the circuit to exhibit stable oscillation, damped oscillation, or stable coexistence.

3*. Rock-scissor-paper game. This classic "game" involves 3 species, R, S, and P interacting in a population. S stimulates the growth of R while P stimulates the death of R. Also, P stimulates the growth of S while R stimulates the death of S, and R stimulates the growth P while S stimulates the death of P. Let p_1, p_2, p_3 denote respectively the frequency of R,S,P in a population, with $p_1 + p_2 + p_3 = 1$. In the simplest case where the gain (cost) of winning (losing) is unity, the dynamics of the system is governed by the following ODEs:

$$\frac{d}{dt}p_1 = p_1 \cdot (p_2 - p_3)$$
$$\frac{d}{dt}p_2 = p_2 \cdot (p_3 - p_1)$$
$$\frac{d}{dt}p_3 = p_3 \cdot (p_1 - p_2)$$

In this problem, you will work out the conditions under which the R-S-P game sustains oscillation.

(a) Show that the above equations admit a *conserved quantity*, $p_1 \cdot p_2 \cdot p_3 = C$, where C is a positive constant fixed by the initial condition, i.e., $C = p_1(0) \cdot p_2(0) \cdot p_3(0)$.

(b) Introducing $x_i = p_i - 1/3$ to describe deviation from the symmetric point $p_1 = p_2 = p_3 = 1/3$, write down the two constraints on p_i (on their sum and product) in terms of x_i. Further introducing $y = p_2 - p_3$, write down the constraint $p_1 \cdot p_2 \cdot p_3 = C$ in terms of x_1 and y. (We will change x_1 to x below to further simplify notation.)

(c) From the constraint on x and y obtained in **(b)**, show that there is a unique maximum for $C(x, y)$ in the allowed space $0 < p_i < 1$. What is the value $C_0 \equiv C(x_0, y_0)$ at the maximum? and what does the location of the maximum (x_0, y_0) corresponds to in terms of the frequencies p_i? Show that for $C \lesssim C_0$ (i.e., for $0 < C_0 - C \ll C_0$), the stable orbits are ellipses centered at (x_0, y_0), i.e., of the form $a(x - x_0)^2 + b(y - y_0)^2 = c$. How does the size of the ellipse depend on the value of $C_0 - C$?

(d) For an arbitrary value of the constant C in the range $0 < C < C_0$, show that $x(t)$ is bounded in the range $x_{\min}(C)$ and $x_{\max}(C)$ (in the sense that for the ellipse in **(c)**, $x(t)$ is bounded between $\pm\sqrt{c/a}$). Find the values of $x_{\min}$ and $x_{\max}$ in the limit $C \to 0$ and show that the trajectory is composed of 3 straight-line segments in this limit. Express these segments in the original variables (p_1, p_2, p_3) and explain in words what is happening along each trajectory. Sketch this trajectory in the (x, y) space, along with the direction of the dynamics. Add to the plot the ellipse in **(c)** obtained in the limit $C \to C_0$. Finally, sketch your guess of what the trajectories should look like for intermediate values of C.

(e) Bonus for the more mathematically inclined: Find the period of oscillation for $C \to 0$. You should be able to your answer in term of $\ln(1/C)$. [Hint: You will need to obtain deviation of the trajectories from the straight-line segments, and most importantly, the turning points for small but non-zero C. As the 3 pieces are symmetrical, you just need to work out one of them.]