Course overview — Spatiotemporal Biodynamics

Part I: population dynamics and ecology

- no space, no genetics, no environment
- effect of replication and death (predation)
- effect of inter-species interaction
  (coop vs competitive, no mechanism)

→ outcome: growth, extinction, oscillatory, phase transition

→ math: coupled ODE, stability analysis,
  phase diagram

Part II: Consumer-Resource Dynamics

- still no space; but explicitly include resource
- genetics: e.g., organism with diff nutrient preference
- trophism: organism at diff. levels of food chain

→ biological issues: competition, cheating, symbiosis

Part III: Spatiotemporal dynamics

- range expansion and chemotaxis
- spatial patterns in development

→ PDEs; mathematics more involved; team projects

Part I: Population dynamics + ecology

\# individuals of species i in a population: N_i

pop. density: $\rho_i \equiv N_i / V$

→ this course: ignore discrete nature of N_i
treat ρ_i as a continuous variable

[effect of demographic noise important for evolution dynamics & certain ecological processes, e.g. invasion; requires stochastic dynamics (PHY 210B)]

A. Intro to pop dynamics

1. Logistic model of pop. growth

- individuals replicate at rate r; no death

$$\frac{d\rho}{dt} = r\rho; \qquad \rho(t) = \rho_0 e^{rt} \to \infty$$

ρ_0: init density at $t=0$

- carrying capacity $\tilde{\rho}$ (common notation: K)

$$\frac{d\rho}{dt} = r\rho \cdot (1 - \rho/\tilde{\rho}) \quad \text{— logistic eqn}$$

→ Simplest eqn to produce the phenomenology that pop grows and saturates: $\frac{d\rho}{dt} \to 0$ as $\rho \to \tilde{\rho}$

a phenomenological description of the effect of starvation / crowding

Warning: this is a phenomenological model; be careful about making mechanistic interpretation!

(e.g. does not describe bacterial growth in batch; will be discussed in detail in Part II)

* Asymptotic approach to final steady state:

let $\rho(t) = \rho^* + \delta\rho(t)$; ρ^* is notation for $\rho(t=\infty) = \tilde{\rho}$

$$\frac{d}{dt}\delta\rho = r(\tilde{\rho} + \delta\rho)\left(\cancel{1} - (\cancel{\tilde{\rho}} + \delta\rho)/\tilde{\rho}\right) = -r\,\delta\rho$$

$\delta\rho(t) \propto e^{-rt}$ same time scale of approach from above + below.

* time to approach steady state from init condition?

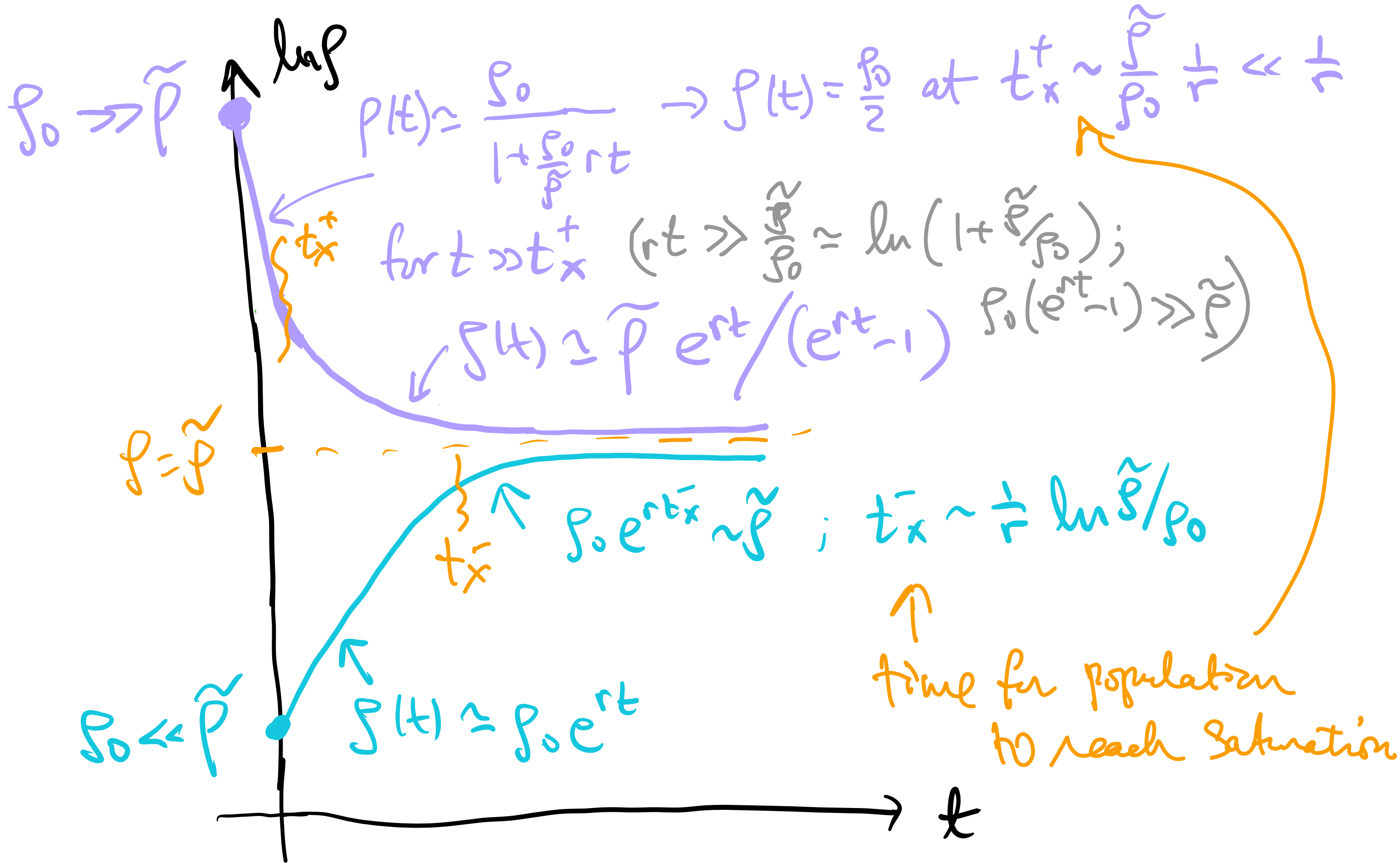
Exact soln: $r\,dt = \dfrac{d\rho}{\rho(1-\rho/\tilde{\rho})}$

$$rt = \int_{\rho_0}^{\rho(t)} d\rho \left[\frac{1}{\rho} + \frac{1/\tilde{\rho}}{1-\rho/\tilde{\rho}} \right]$$

$$= \Big[\ln\rho - \ln(1-\rho/\tilde{\rho})\Big]_{\rho_0}^{\rho(t)} = \ln\left(\frac{\rho}{1-\rho/\tilde{\rho}}\right)\Bigg|_{\rho_0}^{\rho(t)}$$

$$= \ln\left[\left(\frac{\rho(t)}{1-\rho(t)/\tilde{\rho}}\right) \Big/ \left(\frac{\rho_0}{1-\rho_0/\tilde{\rho}}\right)\right]$$

$$\rightarrow \rho(t) = \frac{\rho_0 e^{rt}}{1+\frac{\rho_0}{\tilde\rho}\left(e^{rt}-1\right)} = \begin{cases}\rho_0 & t=0\\ \tilde\rho & t=\infty\end{cases}$$



$\rightarrow$ Why is the time scale for approaching ρ^* (t_x^+ vs t_x^-) so different from above and below?

e.g. for $\rho_0/\tilde\rho = 10^{-3}$, $t_x^- = \frac{1}{r}\ln \tilde\rho/\rho_0 = 7/r$

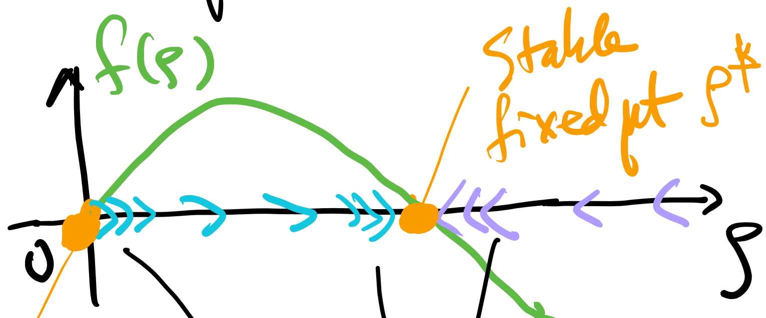
for $\rho_0/\tilde\rho = 10^{3}$; $t_x^+ = \frac{1}{r}\tilde\rho/\rho_0 = \frac{1}{10^3 r}$

– Algebraically, for $\rho_0 \ll \tilde\rho$, $\frac{d\rho}{dt} \simeq r\rho$

for $\rho_0 \gg \tilde\rho$, $\frac{d\rho}{dt} \simeq -r\rho\cdot\overbrace{(\rho_0/\tilde\rho)}^{\gg 1}$

- Alternative: "visual inspection" of the ODE.

$$\frac{d\rho}{dt} = \underbrace{r\rho(1-\rho/\tilde{\rho})}_{f(\rho)}$$



"fixed point" ρ^*: $f(\rho^*)=0$

→ long time from below reflect time to "escape" from fixed point at $\rho=0$.

"visual inspection" important & useful because in most cases $\rho(t)$ cannot be solved analytically.

2. replication + predation

Include the effect of pop loss into logistic growth

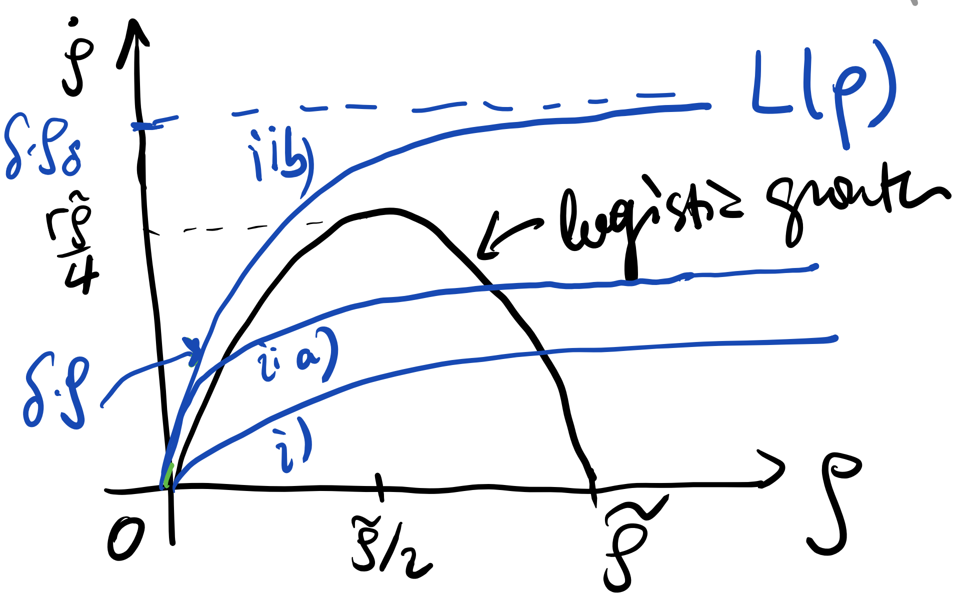
$$\frac{d\rho}{dt} = r\rho\left(1-\frac{\rho}{\tilde{\rho}}\right) - L(\rho)$$

- constant death rate: shifts reprod. rate r.
- effect of predation generally density-dependent
  e.g., killing of bacteria by phage or eukaryote

max loss rate

$$L(\rho) = \frac{\delta\rho}{1+\rho/\rho_\delta}$$

(note: effect of bacteria on predator ignored here)



Case i) $\delta < r$ $\quad \tilde{\rho} > \rho^* > 0$

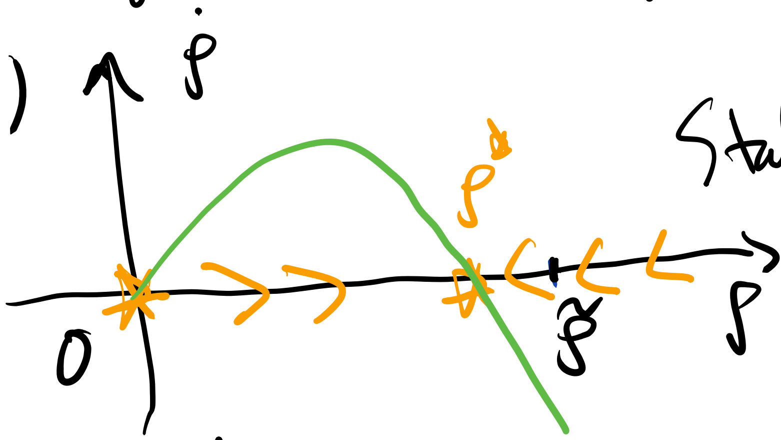
ii) $\delta > r$ <

ii a) $\delta \cdot \rho_\delta \lesssim r\tilde{\rho}/4$ $\quad \rho^* = ?$

ii b) $\delta \cdot \rho_\delta \gtrsim r\tilde{\rho}/4$ $\quad \rho^* = 0$

a) graphical analysis

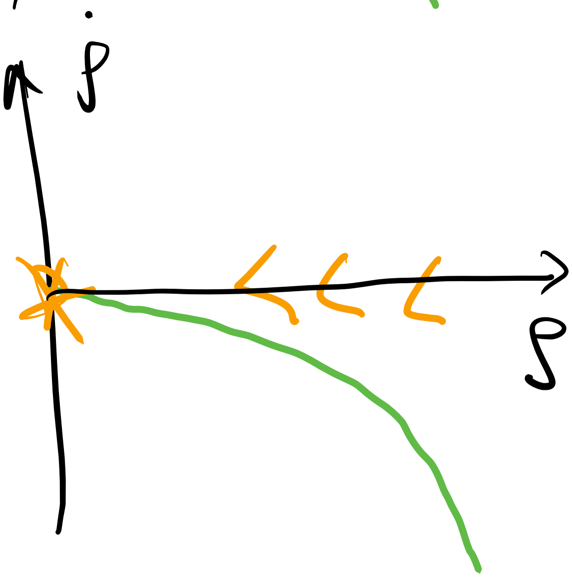
i)



Stable fixed pt at $\rho^* \lesssim \tilde{\rho}$

carrying capacity moderately reduced.

iib)



Pop always driven to extinction

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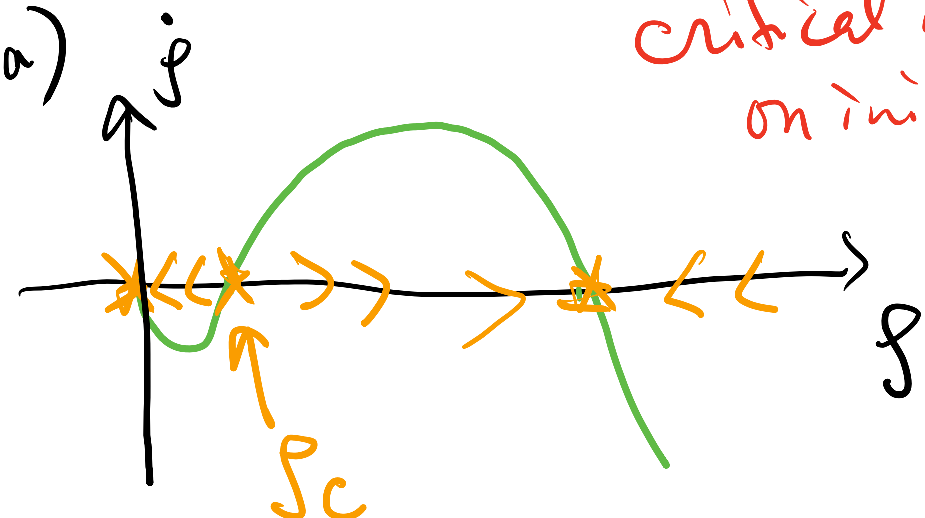

critical dependence on init density ρ_0

"Allee Effect"

two phases:

if $\rho_0 < \rho_c$, then $\rho(t \to \infty) \to 0$ (extinction)

if $\rho_0 > \rho_c$, then $\rho(t \to \infty) \to \rho^* \lesssim \tilde{\rho}$ (Stable existence)

$\rightarrow$ life-or-death depends on init condition.