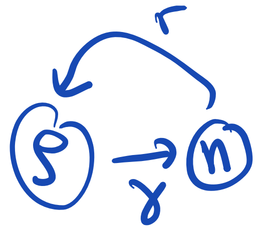
2. Effect of nutrient structure

- So far, nutrients are either supplied or self-generated
- below we study the effect of resource structure where certain species produce nutrients for others

→ cheating & cooperation



a) Nutrient production

- one species of density ρ.
- produces its own nutrient n (e.g., fix CO_2, degrade chitin)

$$\dot\rho = r(n)\rho - \mu\rho, \quad r(n) = r_0 \frac{n}{n+K}$$

$$\dot n = \underbrace{\gamma\rho}_{\text{production; no ext supply}} - \mu n - r(n)\rho/Y$$

γ: nutrient prod. rate

μ: "dilution" rate

($\mu < r_0$ for possibility of existence)

take $\dot n = 0$ (rapid eq of n).

$$\rho = \frac{\mu n}{\gamma - r(n)/Y}$$

two cases:

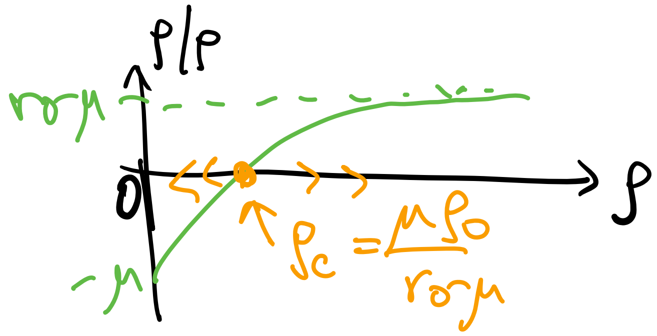
i) $\gamma \gg r_0/Y$ (bottleneck = uptake)

$$\rho \simeq \frac{\mu n}{\gamma} \to \boxed{n(\rho) = \frac{\gamma\rho}{\mu}}$$

$\mu K/\gamma$

$$\to \dot\rho = r(n)\rho - \mu\rho = \left[r_0 \frac{\gamma\rho/\mu}{\frac{\gamma\rho}{\mu}+K} - \mu\right]\cdot\rho = \left[r_0 \frac{\rho}{\rho+\rho_0} - \mu\right]\cdot\rho$$

Allee effect.



- $\rho(0) < \rho_c$, we have $\rho^* \to 0$
- $\rho(0) > \rho_c$, ρ increases s.t.

$$\dot{\rho} \simeq \lambda \rho \; ; \quad \boxed{\lambda = r_0 - \mu}$$

uptake as bottleneck

$$\rightarrow \rho(t) \propto e^{(r_0 - \mu)t}$$

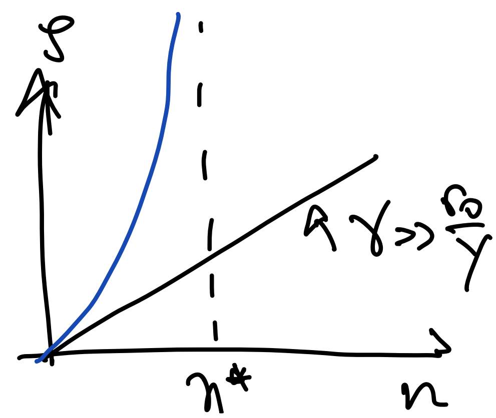
$\Rightarrow$ even at high production rate γ, growth of population occurs only for sufficiently large init. pop size.

Why: nutrient production requires ρ $(n \propto \rho)$

So $\dot{\rho} \propto r(n)\rho \propto \rho^2 < \mu \rho$

ii) $\gamma \ll r_0/Y$ (bottleneck = production)

again, $\dot{n} = 0 \to \rho = \frac{\mu n}{\gamma - r(n)/Y}$



ρ diverges as $\boxed{r(n^*) = \gamma Y}$

Conc where production = Consumption

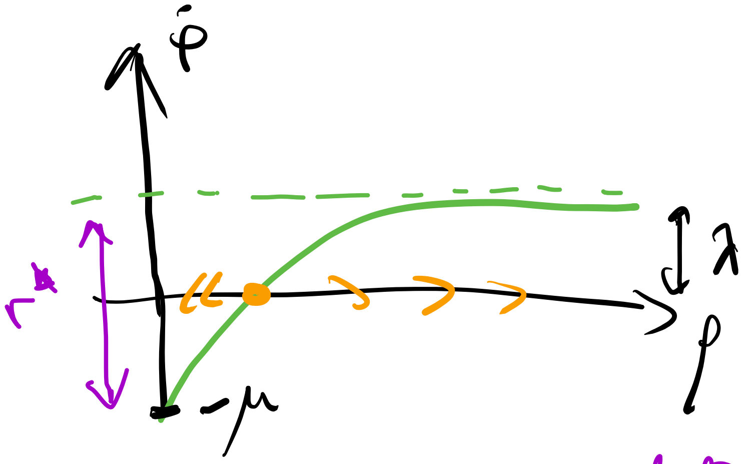
$$r_0 \frac{n^*}{n^* + K} = \gamma Y, \quad \frac{\gamma Y}{r_0} \ll 1 \rightarrow n^*/K \ll 1$$

$$r(n) \simeq r_0 n/K.$$

$$\dot{n} = 0 \rightarrow \gamma \rho = \mu n + r(n)\rho/Y = \left(\mu + \frac{r_0}{KY}\rho\right) n$$

$\rightarrow \; n = \dfrac{\gamma\rho}{\mu + \frac{r_0}{KY}\rho} = \dfrac{\gamma K Y}{r_0}\,\dfrac{\rho}{\rho+\rho_1}$; $\rho_1 = \dfrac{\mu K Y}{r_0}$

$$\dot\rho = \left[\frac{r_0}{K}\, n(\rho) - \mu\right]\rho$$
$$= \left(\underbrace{r^*}_{\gamma Y}\, \frac{\rho}{\rho+\rho_1} - \mu\right)\rho$$



for $\rho(0) > \rho_c$,

- $\rho(t) \propto e^{\lambda t}$, where $\boxed{\lambda = r^* - \mu = \gamma Y - \mu}$.

production as bottleneck

$\Rightarrow$ pop survives only if $\gamma Y > \mu$.

Note 1: batch culture growth; Allee effect manifested as a init conc-dependent lag period. HW

Note 2: γ treated so far as fixed quantity. it is actually regulated and GR dependent

$n^* = \gamma Y K / r_0$ set by cell

increasing γ increases n^*

$\rightarrow$ increases growth but also invite "cheaters"

b) nutrient leakage:

- So far, $\frac{dn}{dt} = \dots - \mu n$
  where the last term representing nutrient loss from dilution in chemostat
- nutrient loss can also occur in nature in systems with spatial compartmentalization

Case study: chitin degradation [Guessous et al, 2023]

- chitin = polymer of N-acetyl glucosamine (GlcNAc)
  one of the most abundant biopolymers in nature
  (e.g. major component of shell of crustaceans)
- chitin is not dissolvable in water, forming "particles"
- chitin breakdown requires specialized enzymes "chitinases"
- chitinase produced by many Vibrio species
- Vibrio cells and chitinase preferentially stick to chitin

Consider batch culture of Vibrio cells with chitin particles as the sole carbon source

- chitin particles: spheres of radius R_0;
  # particles: N_c; particle density $\rho_c = N_c / V$ culture volume
  (particles large so that R_0, $\rho_{chitin} \simeq$ constant)
- Vibrio cells: $N_s(t)$ cells attached to surface
  $N_b(t)$ cells in the bulk (planktonic)

density of cells on chitin particle

$$\sigma(t) = \frac{N_s(t)/N_c}{4\pi R_0^2}$$

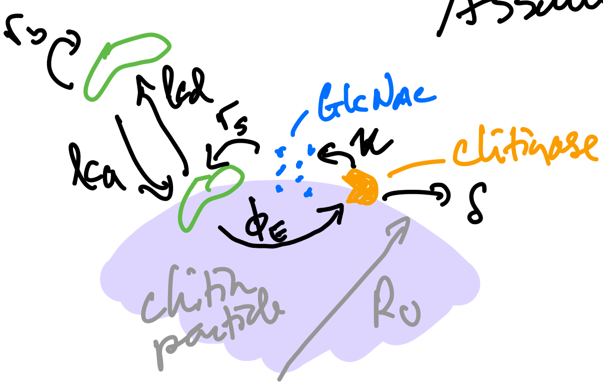
(assuming uniform dist. on particle surface)

- chitinase: enzyme density on particle surface $\mathcal{E}(t)$

Number of surface attached enzymes:

$$N_E(t) = \mathcal{E}(t)\cdot 4\pi R_0^2 \cdot N_c.$$

Zoom into dynamics on particle surface



Assume exponential growth: detachment rate

$$\frac{d}{dt}N_s = r_s N_s - k_d N_s + k_a N_b$$

$$\frac{d}{dt}N_b = r_b N_b - k_a N_b + k_d N_s$$

attachment rate

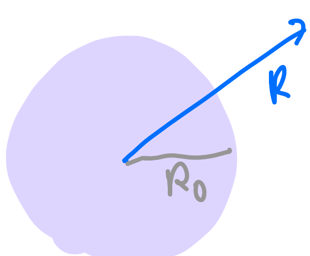
depend on local nutrient conc $n(R,t)$ $\rightarrow$
r_s : replication rate on particle
r_b : rep rate in the bulk

enzyme loss or turnover

enzyme production : $\frac{d}{dt}N_E = \phi_E \cdot r_s N_s - \delta N_E$

fraction of protein synthesis towards chitinase

nutrient dynamics : $n(R,t)$



enzyme production rate = # GlcNAc / chitinase

$$\frac{\partial}{\partial t}n = D\nabla^2 n + \left(\kappa\cdot\mathcal{E} - \frac{r_s\sigma}{Y}\right)\delta(R-R_0)$$

(will see uptake by bulk cells negligible)

- Seek "steady state" nutrient distribution for "frozen" ε and σ (much slower time scale)

$\rightarrow$ $n^*(R) \propto \frac{1}{R}$ "Coulomb potential"

(for spherical shell of "surface charge")

let nutrient conc at surface be $n^*(R_0) \equiv n_s$

then $n^*(R) = n_s \frac{R_0}{R}$

$\rightarrow$ drop in $n^*(R)$ away from particle implies $r_b \propto n^*(R) \rightarrow 0$

(n(R) drops even faster if consumption by N_b included)

- balance of nutrient flux at surface:

$$\kappa \cdot \varepsilon - \frac{r_s \sigma}{Y} = -D \cdot \underbrace{\left.\frac{\partial n^*}{\partial R}\right|_{R_s}}_{= -n_s/R_0}$$

$$\Rightarrow \underset{\substack{\uparrow \\ \text{nutrient} \\ \text{production}}}{\kappa \cdot \varepsilon} = \underset{\substack{\uparrow \\ \text{nutrient} \\ \text{consumption}}}{\frac{r_s \sigma}{Y}} + \underset{\substack{\uparrow \\ \text{nutrient loss} \\ \text{due to diffusion}}}{\frac{D}{R_0} n_s}$$

take $r_s = r_0 \frac{n_s}{n_s + K} \simeq \overset{r_0/K}{\nu} \cdot n_s$

then $\kappa \cdot \varepsilon = n_s \cdot \left(\frac{\nu \sigma}{Y} + D/R_0 \right)$

back to eqn for N_s, N_b, E.

$$\begin{cases} \frac{d}{dt} N_s = \nu n_s N_s - k_d N_s + k_a N_b \\ \frac{d}{dt} N_b = k_d N_s - k_a N_b \\ \frac{d}{dt} N_E = \phi_E V n_s N_s - \delta E \end{cases}$$

$$n_s = \frac{\kappa \cdot \mathcal{E}(t)}{\frac{\nu}{Y}\sigma(t) + D/R_0} = \frac{\kappa N_E(t)}{\frac{\nu}{Y}[N_s(t) + N_0]}$$

$$\sigma(t) = \frac{N_s(t)/N_c}{4\pi R_0^2}$$

$$\mathcal{E}(t) = \frac{N_E(t)/N_c}{4\pi R_0^2}$$

where $N_0 \equiv \frac{D/R_0}{\nu/Y} \cdot N_c 4\pi R_0^2 = \frac{D/R_0^2}{\nu/Y} 3 \rho_c V$

$$\rho_c = \frac{4\pi}{3} R_0^3 N_c / V$$
$= $ chitin particle density

$\frac{N_0}{V}$ = cell density whose nutrient uptake is equivalent to diffusive loss

Note 1: determined by the ratio of two time scales
- diffusive loss: D/R_0^2
- nutrient uptake: ν/Y

Note 2: for fixed chitin particle density smaller particles give larger leakage insert $n_s(N_s, N_E)$ into eqn for N_s, N_E: 10b

$$\begin{cases} \frac{d}{dt} N_s = \varkappa Y N_E \cdot \frac{N_s}{N_s + N_0} - k_d N_s + k_a N_b & (1) \\ \frac{d}{dt} N_E = \phi_E \cdot \varkappa Y N_E \cdot \frac{N_s}{N_s + N_0} - \delta N_E & (2) \\ \frac{d}{dt} N_b = k_d N_s - k_a N_b. & (3) \end{cases}$$

* exponential growth phase: $N_{s,b,E}(t) \propto e^{\lambda t}$

$$\frac{N_s}{N_s + N_0} \to 1$$

Eq 2 becomes: $\lambda N_E = (\phi_E \varkappa Y - \delta) N_E$

$$\to \boxed{\lambda = \phi_E \varkappa Y - \delta}$$

pop. growth rate set by chitinase production/activity

↑ growth bottleneck

Eq 1 + 3: $\lambda (N_s + N_b) = \varkappa Y N_E = (\lambda + \delta) \cdot \frac{N_E}{\phi_E}$

$$\Rightarrow \frac{N_E}{N_s + N_b} = \phi_E \cdot \frac{\lambda}{\lambda + \delta}$$

(enzyme fraction set by ϕ_E up to turnover)

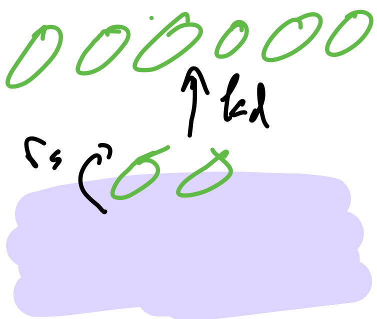
Eq 3: $\lambda N_b = -k_a N_b + k_d N_s$

$$\to \frac{N_b}{N_s} = \frac{k_d}{\lambda + k_a} \; ; \; \frac{N_b}{N_b + N_s} = \frac{k_d}{\lambda + k_d + k_a}$$

Measurements: $\lambda \simeq 0.06/h$, $k_d \simeq 0.18/h$

$\frac{N_b}{N_b + N_s} \simeq 0.75 \rightarrow k_a \ll \lambda$ (negligible)

revisit eqn 1: $\frac{d}{dt} N_s = \underbrace{\kappa Y N_E \frac{N_s}{N_s + N_0}}_{r_s} - k_d N_s$



$\Rightarrow \boxed{\lambda = r_s - k_d}$

$\rightarrow r_s = \lambda + k_d \simeq 0.24/h$

Note: increasing k_d increases r_s, does not reduce λ

λ set by chitinase production ϕ_E.

(This can happen because all chitinase are left on the particle to feed attached cells)

* approach to exponential growth (Allee effect?)

for $k_a \simeq 0$, just need to follow N_s, N_E.

$$\begin{cases} \frac{d}{dt} N_s = \kappa Y N_E \cdot \frac{N_s}{N_s + N_0} - k_d N_s \\ \frac{d}{dt} N_E = \phi_E \cdot \kappa Y N_E \cdot \frac{N_s}{N_s + N_0} - \delta N_E \end{cases}$$

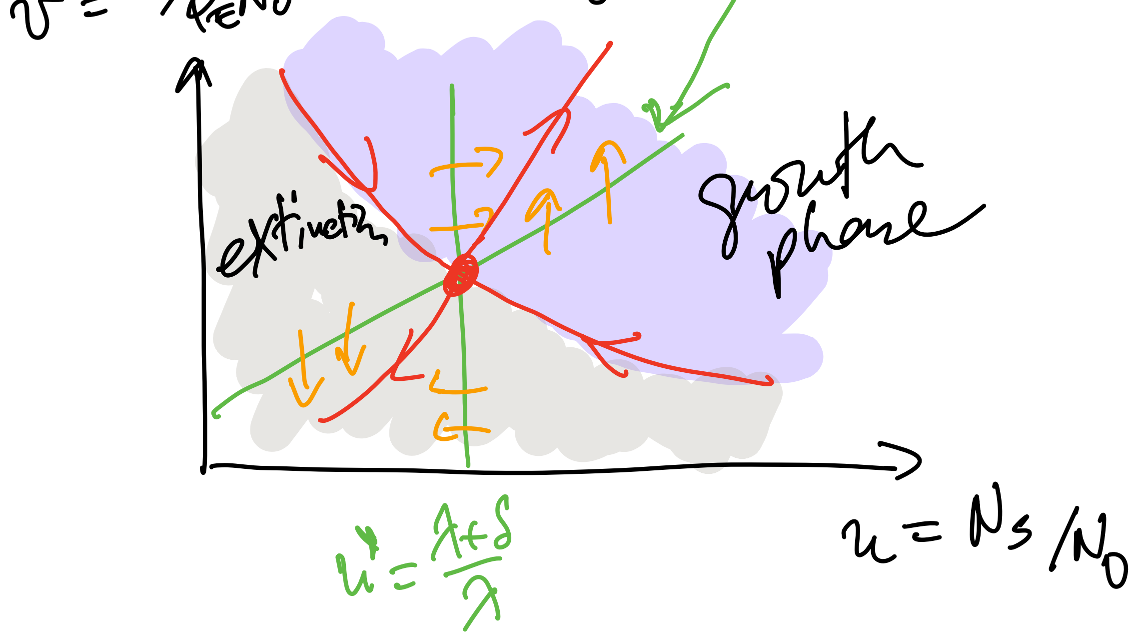
make dimensionless: $u = \frac{N_s}{N_0}$, $v = \frac{N_E}{\phi_E N_0}$, $\tau = \underbrace{\phi_E \times Y}_{=\lambda+\delta} t$ 108

$$\dot{u} = v \cdot \frac{u}{u+1} - \frac{k_d}{\lambda+\delta} u$$

$$\dot{v} = v \frac{u}{u+1} - \frac{\delta}{\lambda+\delta} v$$

null clines: $\frac{\dot{u}}{u} = 0$ $\quad v = \frac{k_d}{\lambda+\delta}(u+1)$

$\frac{\dot{v}}{v} = 0$ $\quad u = \frac{\delta}{\lambda+\delta}(u+1)$



- Allee effect due to nutrient loss (and cell/enzyme detachment)

- init enzyme amt can determine the fate of the culture

(init enzyme amt = amt synthesized by cells before obtaining nutrient from chitin, i.e., enzyme synthesized in stationary phase)