b) Analytical study: quantitative dependence of dynamics on system parameters

$$\frac{d\rho}{dt} = r\rho\left(1-\frac{\rho}{\tilde{\rho}}\right) - \frac{\delta\cdot\rho}{1+\rho/\rho_s}$$

→ make dimensionless: $u=\frac{\rho}{\tilde{\rho}}$, $rt=\tau$, $\frac{\delta}{r}\equiv\alpha$, $\frac{\rho_s}{\tilde{\rho}}\equiv\kappa$

$$\frac{du}{d\tau} = u(1-u) - \frac{\alpha u}{1+u/\kappa}$$

two dimensionless parameters:

$\alpha=\frac{\delta}{r}$ large α = high predation rate

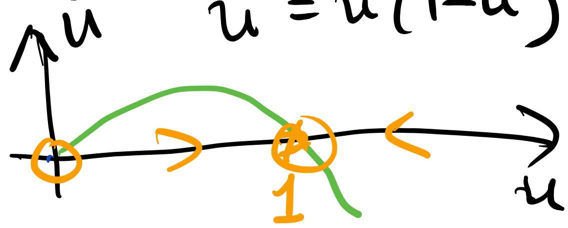
$\kappa=\rho_s/\tilde{\rho}$ large κ = large predation capacity

Steady-state:

extinction existence (if $u^*>0$)

$$\frac{du}{d\tau}=0 \rightarrow u^*=0 \text{ or } 1-u^* = \frac{\alpha}{1+u^*/\kappa}$$

$\alpha=0$. $u^*=1$ $\dot{u}=u(1-u)$

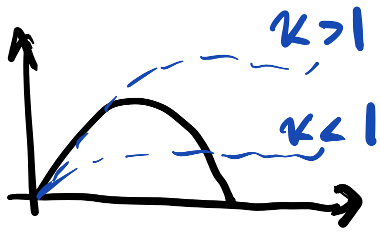


$\alpha>0$ $(1-u^*)\cdot(\kappa+u^*)=\alpha\kappa$

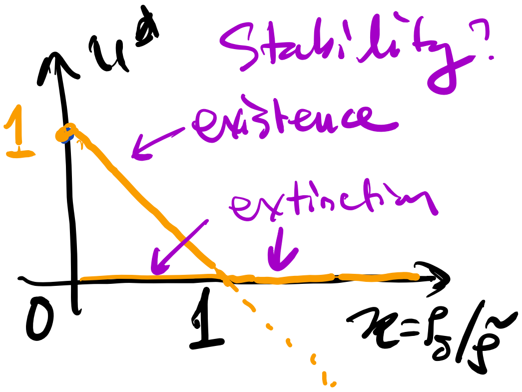
$$(u^*)^2+(\kappa-1)u^*+(\alpha-1)\kappa=0$$

$\alpha = 1$

borderline case with $\delta = r$



$$\boxed{u^* = 0,\ 1-\kappa}$$

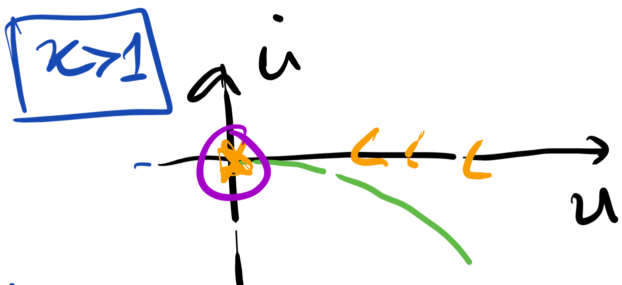


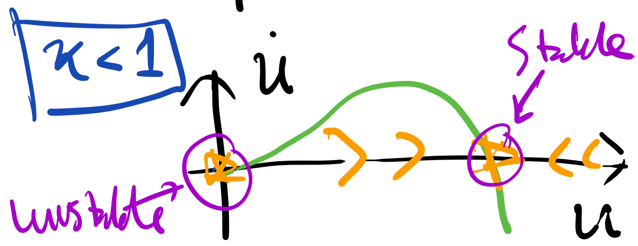
$$\dot u = u(1-u) - \frac{u}{1+u/\kappa}$$

$$= \frac{u}{1+u/\kappa}\cdot\left[(1-u)(1+u/\kappa)-1\right]$$

$$= \frac{u}{1+u/\kappa}\left[\frac{u}{\kappa}(1-u)-u\right]$$

$$= \frac{u^2}{u+\kappa}\left[1-\kappa-u\right]$$

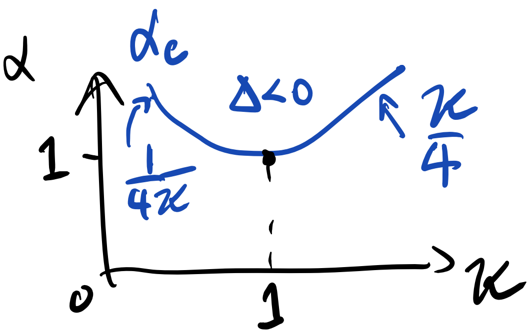




c) phase diagram : which phase is exhibited for what parameters (α, κ)

$$u^* = \frac{1-\kappa}{2} \pm \sqrt{\underbrace{\left(\frac{\kappa-1}{2}\right)^2 - (\alpha-1)\kappa}_{\Delta\ \text{(discriminant)}}} > 0$$

* $\Delta = 0 \rightarrow \kappa^2 + 2\kappa + 1 = 4\alpha_c\kappa$



$$4\alpha_c = \kappa + 2 + \frac{1}{\kappa}.$$

$$4\alpha_c' = 1 - \frac{1}{\kappa^2} = 0$$

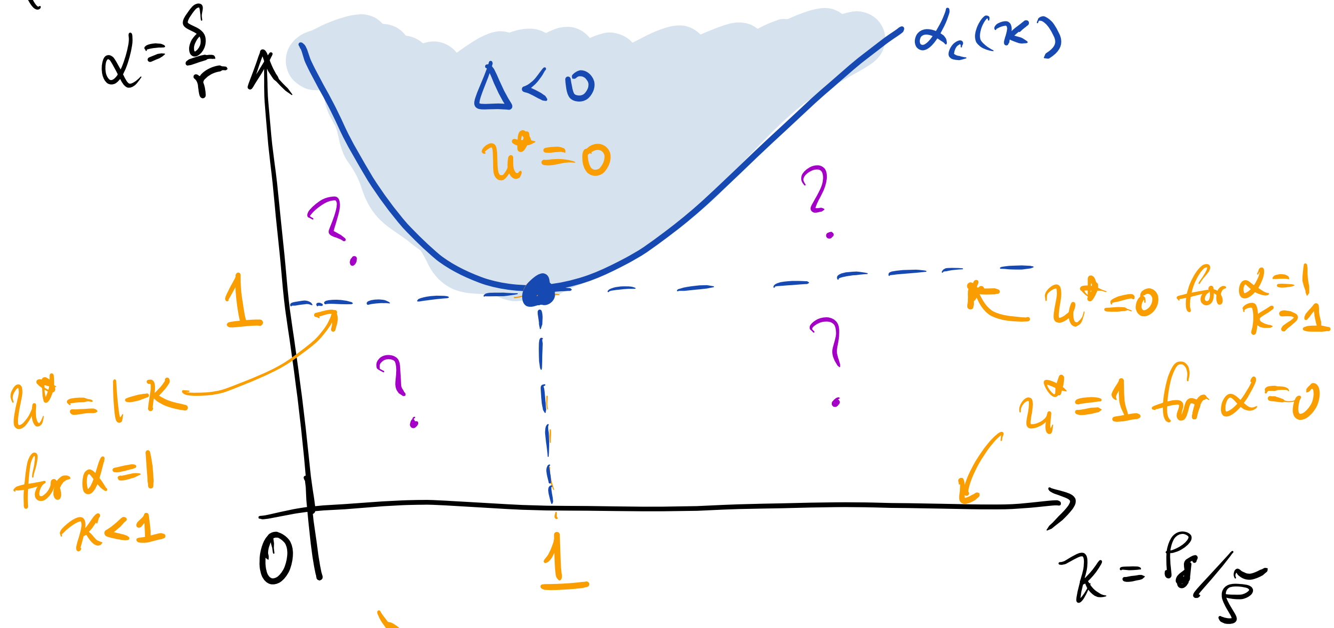
min at $\kappa = 1$, $\alpha_c = 1$ $\left(4\alpha_c'' = \frac{2}{\kappa^3} > 0\right)$.

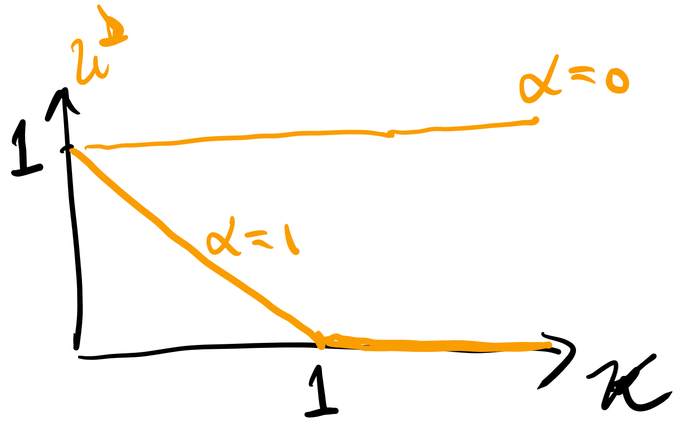
* for $\alpha > \alpha_c$, $\Delta < 0$ → only sol'n is $u^* = 0$ (extinction!)

no real soln

* for $\alpha < \alpha_c$; $\Delta > 0$
→ one or two sol'n with $u^* > 0$ — stable sol'n / phase transition

phase diagram:

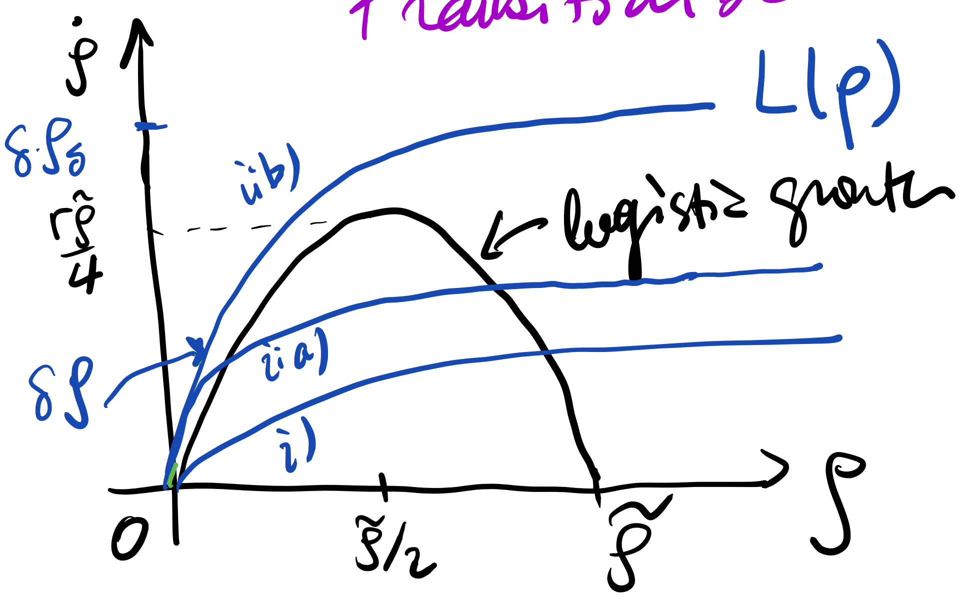




$u^*(\alpha, \kappa)$ for other values of α?

Recall 3 phases discussed at beginning

transition between phases = "bifurcation"



i) $\delta < r$ $(\alpha < 1)$

ii) $\delta > r$ $(\alpha > 1)$

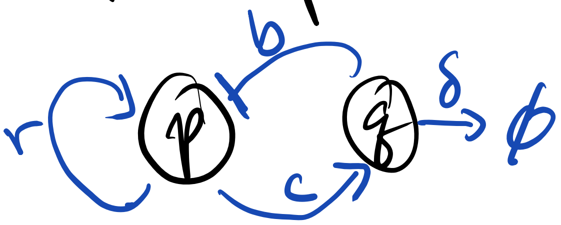
ii a) $\delta \cdot \rho_\delta \lesssim r\tilde{\rho}/4$ $(\alpha < \kappa/4)$

ii b) $\delta \cdot \rho_\delta \gtrsim r\tilde{\rho}/4$ $\left(\alpha > \frac{\kappa}{4}\right)$

3. Two-Species interaction

a) Predator-Prey system

- two species:



prey (density p)
predator (density q)

$$\frac{dp}{dt} = r p - b p q$$
$$\frac{dq}{dt} = c p q - \delta q$$

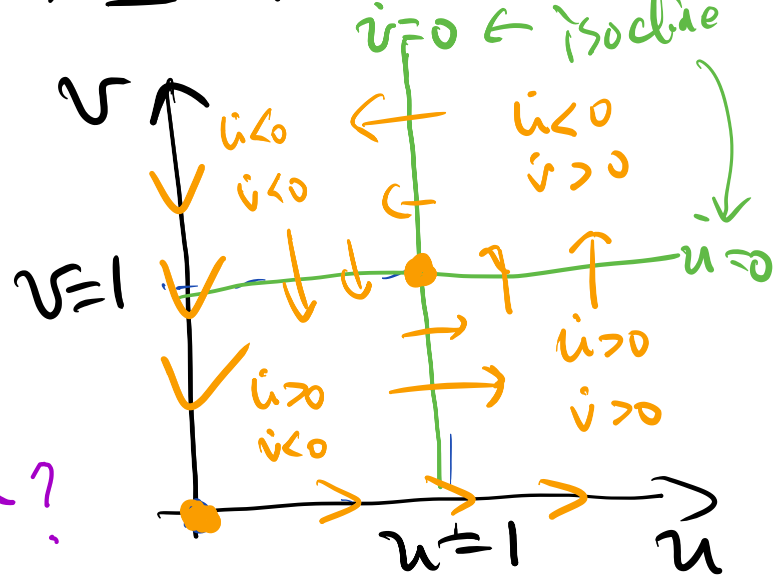
Lotka-Volterra Model

Volterra (1926):
⌈Lotka (1920) introduced same eqn to describe chemical reactions⌋

make dimensionless: $u = p/\frac{\delta}{c}$, $v = q/\frac{r}{b}$, $\tau = r.t$, $\alpha = \delta/r$

$$\begin{cases} \dfrac{du}{d\tau} = u(1-v) \\ \dfrac{dv}{d\tau} = \alpha v (u-1) \end{cases}$$

Sketch of 2d flow



fixed pt: $(u^*, v^*) = (0,0), (1,1)$

→ dynamics around $(1,1)$:
damped or unstable spiral?

let $\begin{cases} u = 1 + x \\ v = 1 + y \end{cases}$ → $\begin{cases} \dfrac{dx}{d\tau} = -y \\ \dfrac{dy}{d\tau} = \alpha x \end{cases}$ or $\dfrac{d^2x}{d\tau^2} = -\dfrac{dy}{d\tau} = -\alpha x$

$$\Rightarrow \begin{cases} x(\tau) = x_0 e^{\pm i\sqrt{\alpha}\tau} \\ y(\tau) = y_0 e^{\mp i\sqrt{\alpha}\tau} \end{cases}$$

oscillatory sol'n; but depends on init cond ??

full solution: $\dfrac{du/d\tau}{dv/d\tau} = \dfrac{du}{dv} = \dfrac{u(1-v)}{\alpha v(u-1)}$

$$\alpha \frac{du}{u}(u-1) = \frac{dv}{v}(1-v)$$

$$\alpha u - \ln u^{\alpha} = \ln v - v + H$$

const.

$$\boxed{H(u,v) = \alpha u + v - \alpha \ln u - \ln v}$$

– conserved quantity !

at $u=1, v=1$, $H(1,1) = 1+\alpha \equiv H_0$

$\rightarrow$ $H(u,v)$ has global min at $u=1, v=1$ (with value $H_0 = 1+\alpha$)

$$\left[\begin{array}{l} \dfrac{\partial H}{\partial u} = \alpha - \dfrac{\alpha}{u} = 0 \text{ at } u^* = 1 ; \quad \dfrac{\partial^2 H}{\partial u^2} = \dfrac{\alpha}{u^2} > 0 \\ \dfrac{\partial H}{\partial v} = 1 - \dfrac{1}{v} = 0 \text{ at } v^* = 1 ; \quad \dfrac{\partial^2 H}{\partial v^2} = \dfrac{1}{v^2} > 0 \end{array} \right]$$

trajectory of orbits :

- $H \gtrsim H_0$: let $u = 1+x$, $v = 1+y$.

$$\alpha + \alpha x + 1 + y - \ln(1+x)^{\alpha} - \ln(1+y) = H$$

$$(\cancel{\alpha x + y}) - \alpha(\cancel{x} - \tfrac{x^2}{2} \cdots) - (\cancel{y} - \tfrac{y^2}{2} \cdots) = H - H_0$$

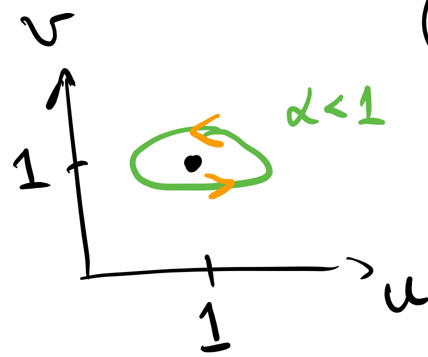
$\rightarrow \alpha \frac{x^2}{2} + \frac{y^2}{2} = H - H_0$ (ellipse)

$y = 0 \quad x = \pm\sqrt{\frac{2}{\alpha}(H - H_0)}$

long-axis along x if $\alpha < 1$

(Corresponding to $\delta < r$)

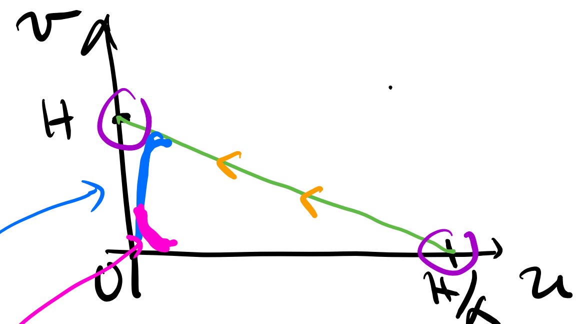
predator death rate (δ), prey replication rate (r)



- $H \gg H_0$: look at $\alpha u + v - \alpha \ln u - \ln v = H$

for large u, v, can neglect $\ln u$, $\ln v$

$\rightarrow \alpha u + v \simeq H$.



- breaks down when $u, v \to 0$

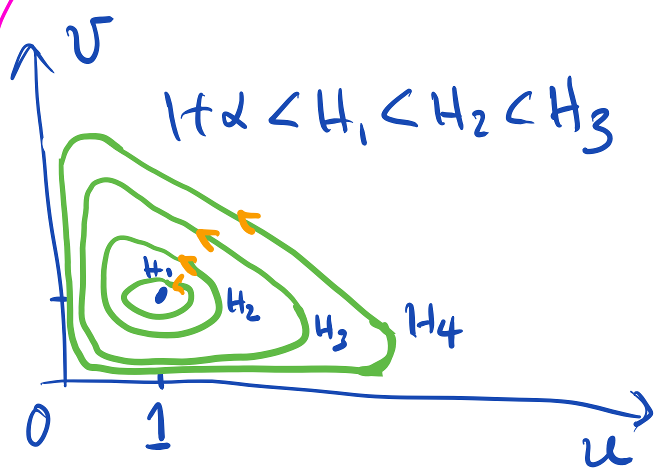
Consider $v \simeq H$, $u \to 0$.

here $v \simeq H + \alpha \ln u$

- for both $u, v \ll 1$

$-H = \alpha \ln u + \ln v = \ln v \cdot u^\alpha$

$v = u^{-\alpha} e^{-H}$



Overall lesson: conserved quantity → periodic orbit.

but solution completely dependent on init cond.

b) break conserved quantity

e.g. include logistic growth of prey.

$$\begin{cases} \frac{dp}{dt} = rp\left(1 - p/\tilde{p}\right) - bpq \\ \frac{dq}{dt} = cpq - \delta q \end{cases}$$

ratio of death rate to max predator growth rate

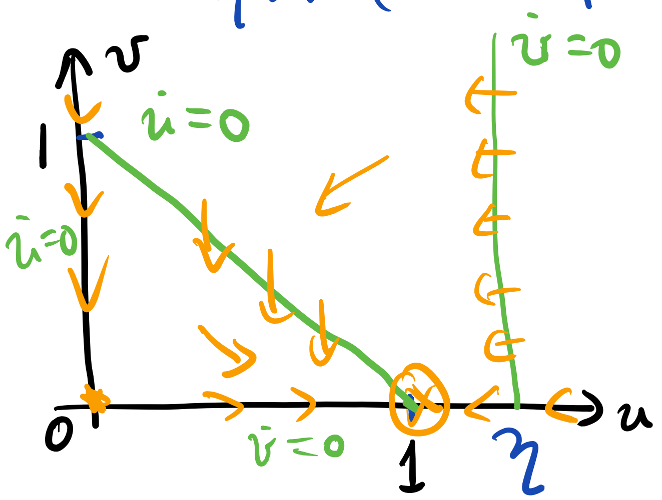
dimensionless: $u = p/\tilde{p}$, $v = q\frac{b}{r}$, $\tau = r \cdot t$, $\alpha = \delta/r$, $\eta = \frac{\delta}{\tilde{p}c}$

$$\begin{cases} \frac{du}{d\tau} = u(1 - u - v) \\ \frac{dv}{d\tau} = \alpha v \cdot \left(\frac{u}{\eta} - 1\right) \end{cases}$$

$\dot{u} = 0$; $u = 0$ or $u + v = 1$

$\dot{v} = 0$: $v = 0$ or $u = \eta$

$\eta > 1$ (or small $\tilde{p}$: small carrying capacity for prey)

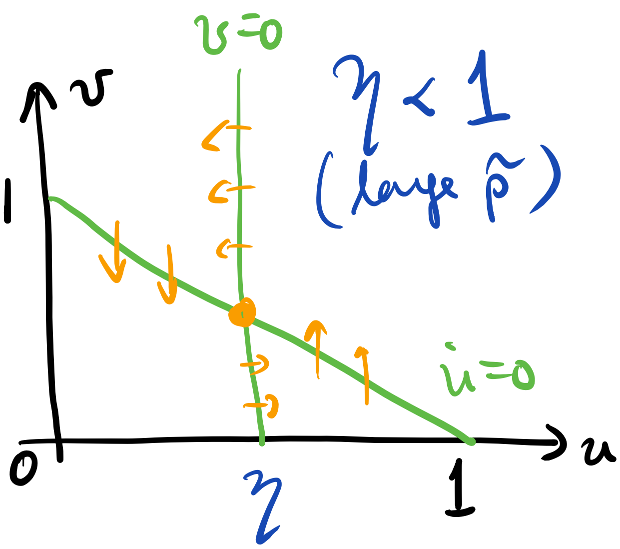


predator extinct

prey → carrying capacity

⇒ Small carrying capacity for prey leads to collapse of predators

(c.f. dinosaur)



Check for stability and oscillation

$u^* = \eta$, $v^* = 1 - u^* = 1 - \eta$

$u = \eta + x$. $v = 1 - \eta + y$.

for small x, y

$$\begin{cases} \frac{dx}{d\tau} = -\eta \cdot (x + y) \\ \frac{dy}{d\tau} = \alpha(1-\eta) \cdot \frac{x}{\eta} \end{cases}$$