$$\frac{d}{d\tau}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -\eta & -\eta \\ \frac{\alpha(1-\eta)}{\eta} & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}; \text{ look for } \begin{cases} x(\tau) = e^{\lambda\tau} \\ y(\tau) = e^{\lambda\tau} \end{cases} \to \begin{bmatrix} -\eta-\lambda & -\eta \\ \frac{\alpha(1-\eta)}{\eta} & -\lambda \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0$$

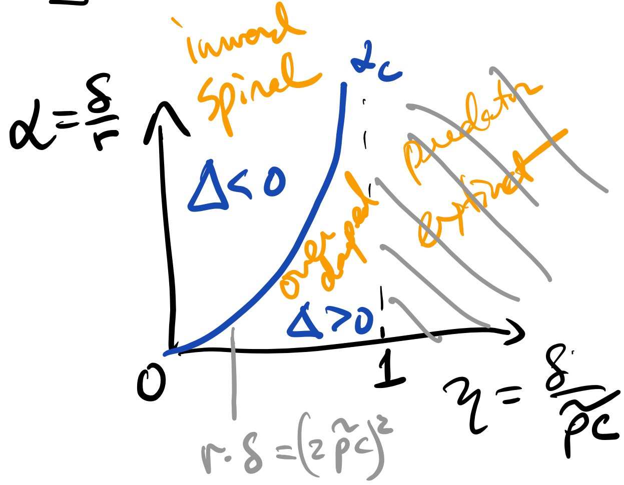
Solve for λ by taking $\det[\quad] = 0$

$$\lambda^2 + \eta\lambda + \alpha(1-\eta) = 0.$$

$$\lambda = -\frac{\eta}{2} \pm \sqrt{\underbrace{\left(\frac{\eta}{2}\right)^2 - \alpha(1-\eta)}_{\Delta}}$$

$$\Delta = \left(\frac{\eta}{2}\right)^2 - \alpha(1-\eta) = 0$$

$$\to \quad \alpha_c = (\eta/2)^2/(1-\eta)$$



if $\Delta < 0$: $\lambda = -\frac{\eta}{2} \pm i\sqrt{|\Delta|}$ — damped oscillation (towards coexistence) (large $\tilde{p}$)

if $\Delta > 0$: $\lambda = -\frac{\eta}{2} \pm \underbrace{\sqrt{\Delta}}_{< \frac{\eta}{2}} < 0$ — over damped (small $\tilde{p}$) (slow approach to coexist)

$\Rightarrow$ stable oscillation exhibited by the simple Lotka-Volterra model (corresponding to $\tilde{p} \to \infty$) is not robust

$\Rightarrow$ existence vs extinction depends only on $\eta = \delta/\tilde{p}c$
occurrence of (damped) oscillation also depend on $\alpha = \delta/r$.

Note: if $\mathrm{Re}\{\lambda\} > 0$ and $\mathrm{Im}\{\lambda\} \neq 0$, and further if u and v are bounded, then obtain stable limit cycle (Poincare-Bendixson Theorem)

$\to$ Will show this occurs when saturation of predation is included

3 c) Epidemic Model: Spread of infection (16)

3 distinct classes of individuals in population:

S: susceptible — SIR model

I: infected and can transmit

R: removed (recovered, immune, isolated or dead)

Assume: uniformly mixed

(every pair of individuals has equal probab of contact)

$$\begin{cases} \frac{dS}{dt} = -rS \cdot I & (1) \\ \frac{dI}{dt} = rSI - \delta I & (2) \\ \frac{dR}{dt} = \delta I & (3) \end{cases}$$

r: infection rate/suscep. (not replication rate)

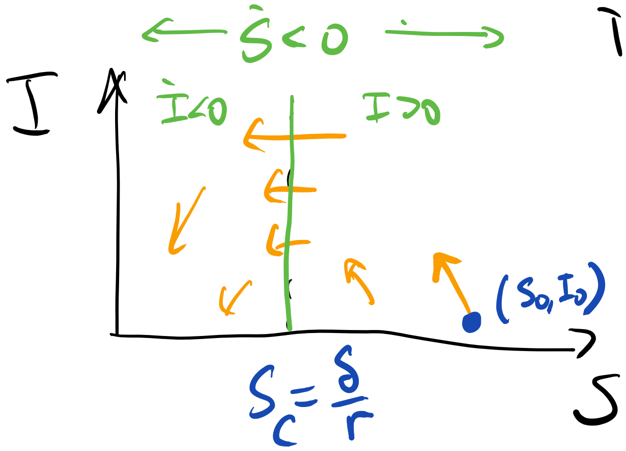
δ: removal rate

Note: $\frac{dS}{dt} + \frac{dI}{dt} + \frac{dR}{dt} = 0$

$\rightarrow S(t) + I(t) + R(t) = N$ ← pop size

(compared to LV: only missing $+S$ term in (1)) ← loss of immunity

focus on the dynamics of S and I:



Init cond: $I(0) = I_0 > 0$, $R(0) = 0$

$S(0) = S_0 \cong N$

Key parameter:

$$\frac{S_0}{S_c} = \frac{S_0 r}{\delta} = r_0$$

replication rate

"basic reproduction number"

from Eq ① + ②: $\frac{dI}{dS} = \frac{dI/dt}{dS/dt} = \frac{rSI - \delta I}{-rSI} = -1 + \frac{\delta}{rS}$

integrate: $I(t) = \int^{S(t)} dS' \left(-1 + \frac{\delta}{rS'}\right) + \text{const}$

$= -S(t) + \frac{\delta}{r} \ln S(t) + \text{const}$

$\Rightarrow \quad I(t) + S(t) - \frac{\delta}{r} \ln S(t) = \cancel{I_0} + S_0 - \frac{\delta}{r} \ln S_0$

negligible ($I_0 \ll S_0$)

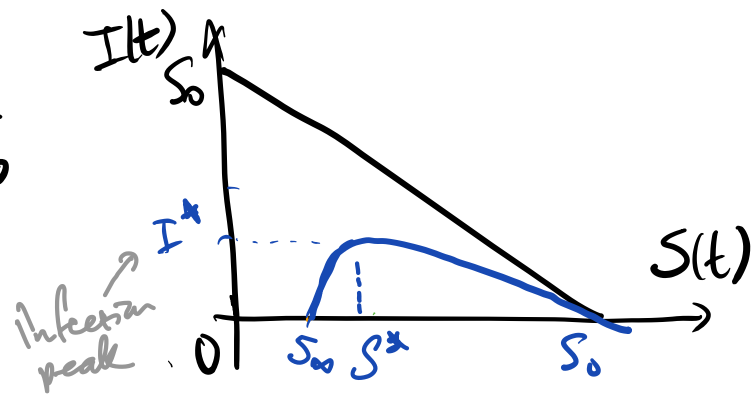
or $\frac{I(t)}{S_0} = 1 - \frac{S(t)}{S_0} + \underbrace{\frac{\delta}{rS_0}}_{1/r_0} \ln S(t)/S_0$

* max infection: $\frac{dI}{dS} = 0$

$\Rightarrow S^* = \frac{\delta}{r}$ or $\frac{S^*}{S_0} = \frac{\delta}{rS_0} = \frac{1}{r_0}$

$\frac{I^*}{S_0} = 1 - \frac{S^*}{S_0} + \frac{S^*}{S_0} \ln S^*/S_0$

$= 1 - \frac{1}{r_0} + \frac{1}{r_0} \ln \frac{1}{r_0}$



if $r_0 = 2.5$; then $\frac{S^*}{S_0} = \frac{1}{2.5} = 40\%$; peak infection $\frac{I^*}{S_0} \approx 23\%$

→ need to infect 60% of pop to acquire "herd" immunity

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as $t \to \infty$, $I \to 0$, $S \to S_\infty$. $S_\infty + R_\infty = S_0$

total infected: $I_{total} = S_0 - S_\infty = R_\infty$

to find S_∞. use Eq ① + ③

$\frac{dS}{dR} = \frac{dS/dt}{dR/dt} = -\frac{rS}{\delta} \longrightarrow S(t) = S_0 e^{-\frac{r}{\delta} R(t)}$

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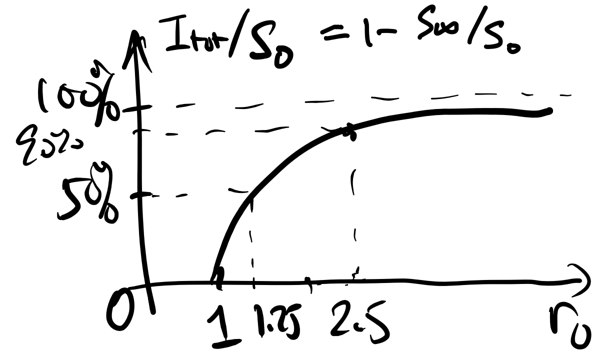
$$S_\infty = S_0 e^{-\frac{r}{\delta} R_\infty} = S_0 e^{-\frac{r}{\delta}(S_0 - S_\infty)} \; ; \quad \frac{S_\infty}{S_0} = e^{-r_0 \left(1 - \frac{S_\infty}{S_0}\right)}$$

$$\frac{I_{total}}{S_0} = 1 - \frac{S_\infty}{S_0} = x. \;\rightarrow\; 1 - x = e^{-r_0 x}$$

- for $r_0 \gtrsim 1$, $r_0 = \frac{-\ln(1-x)}{x} \simeq \frac{x + \frac{x^2}{2} \cdots}{x} = 1 + \frac{x}{2}$

$$x = \frac{I_{tot}}{S_0} \simeq 2 \cdot (r_0 - 1)$$

- for $r_0 \gg 1$. $x \simeq 1 - e^{-r_0}$

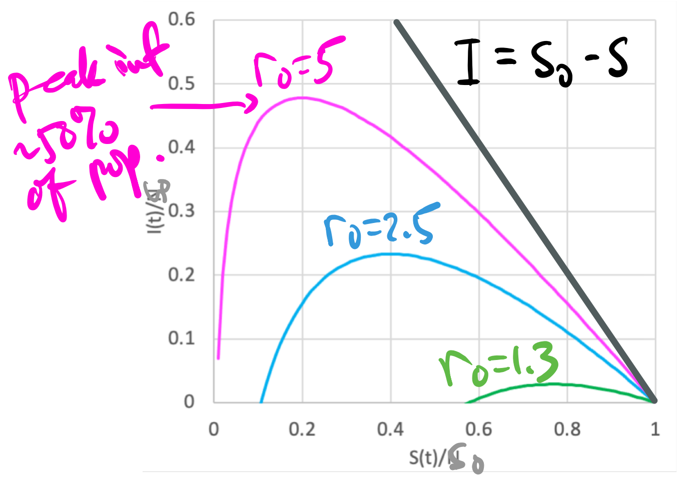


→ $\frac{S_\infty}{S_0}$ not vanishingly small for moderate r_0-values.

(for $r_0 = 2.5$, $\frac{S_\infty}{S_0} = 10\%$
$r_0 = 1.25$ $\frac{S_\infty}{S_0} = 50\%$)

→ Infection stops spreading due to removal, not lack of S. (≃ "herd immunity")

⇒ main effect of reducing r_0 ≃ mitigation is to reduce I^*, not I_{total} (flattening curve)



intervention strategy:

Social distancing: reduce r

rapid detection: increase δ

→ reduces $r_0 = \frac{r S_0}{\delta}$

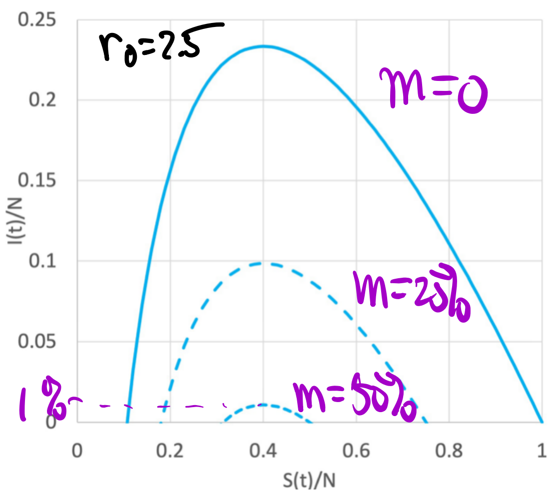
→ flatten the curve!

Another strategy: immunization:

$$I(t) + S(t) - \frac{\delta}{r}\ln S(t) = I_0 + S_0 - \frac{\delta}{r}\ln S_0$$

$S_0 = N\cdot(1-m)$; m = fraction of pop. immunized.

$$\frac{I(t)}{N} = 1 - m - \frac{S(t)}{N} - \frac{\delta}{rN}\ln\frac{S(t)}{N(1-m)}$$



$\frac{rN}{\delta} = r_0 = 2.5$ still.

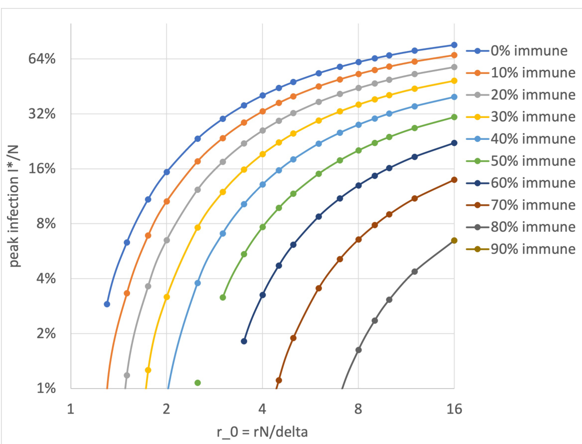
peak still at $S^* = \frac{\delta}{r}$

or $\frac{S^*}{N} = \frac{\delta}{rN} = 40\%$

but pop size effectively reduced from N to $N\cdot(1-m)$

peak infection: $\frac{I^*}{N} = 1 - m - \frac{S^*}{N} - \frac{\delta}{rN}\ln\frac{S^*}{N(1-m)}$

$$= 1 - m - \frac{1}{r_0}\left(1 + \ln[r_0(1-m)]\right)$$

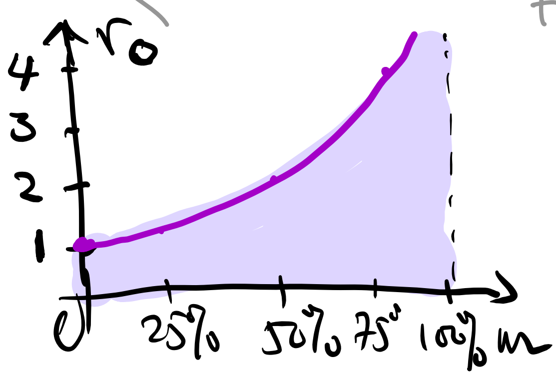


(for $r_0 > \frac{1}{1-m}$)

$\rightarrow$ immunity reached ($I^* = 0$)

when $r_0 = \frac{1}{1-m}$

(need 60% immunization for $r_0 = 2.5$)



★ Kinetics:

– early time: from $\frac{dI}{dt} = rSI - \delta I$

$$\dot{I} \simeq I(t)\cdot(rS_0 - \delta), \quad I(t) \simeq I_0 e^{\overbrace{(rS_0-\delta)}^{\mu=(r_0-1)\cdot\delta} t}$$

estimate of μ gives est of $r_0 = \frac{rS_0}{\delta}$

e.g. if 5 days for symptoms to develop, then $\delta = \frac{\ln 2}{5d}$.

further, if $I(t)/I_0$ doubles every 2.5 days

then at $t=5d$, $\frac{I(5d)}{I(0)} = 4 = e^{(r_0-1)\cdot\delta\cdot 5d} = 2^{(r_0-1)}$

$\rightarrow r_0 = 3$

but estimate of $I(t)$ often unreliable.

– more reliable is $R(t)$: diagnosed and removed.

Eq ③: $$\frac{dR}{dt} = \delta\cdot I(t) = \delta\cdot(N - \overbrace{S(t)}^{S_0 e^{-\frac{r}{\delta}R(t)}} - R(t))$$

$$\frac{dR}{dt} = \delta\cdot\left(N - R - S_0 e^{-\frac{r}{\delta}R}\right)$$

let $z = \frac{R(t)}{N}$, $\tau = \delta\cdot t$, $S_0 = N - I_0 = N(1-\varepsilon)$

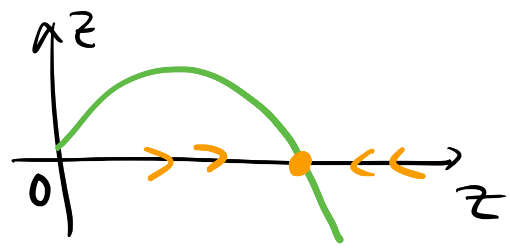
$\varepsilon = I_0/N \simeq 0^+$

$$\frac{I(t)}{N} = \boxed{\frac{dz}{d\tau} = 1 - z - (1-\varepsilon)e^{-r_0 z}}$$

$\rightarrow$ for $r_0 z \ll 1$ (early time or mild epidemics)

$$\dot{z} = \cancel{1} - z - (1-\varepsilon)\left(\cancel{1} - r_0 z + \tfrac{1}{2}(r_0 z)^2\right)$$

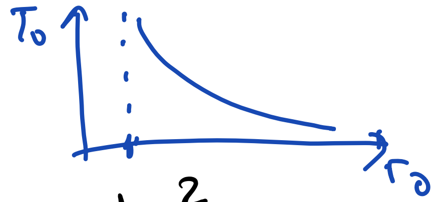
$$= \varepsilon + (r_0 - 1)z - \tfrac{1}{2}(r_0 z)^2$$



Sol'n: (ε→0)

$$\frac{I(\tau)}{N} = \frac{dz}{d\tau} = \frac{1}{2}\left(1-\frac{1}{r_0}\right)^2 \operatorname{sech}^2\left(\frac{r_0-1}{2}(\tau-\tau_0)\right)$$

$$\tau_0 = \frac{2\tanh^{-1}(r_0-1)}{(r_0-1)^2}$$

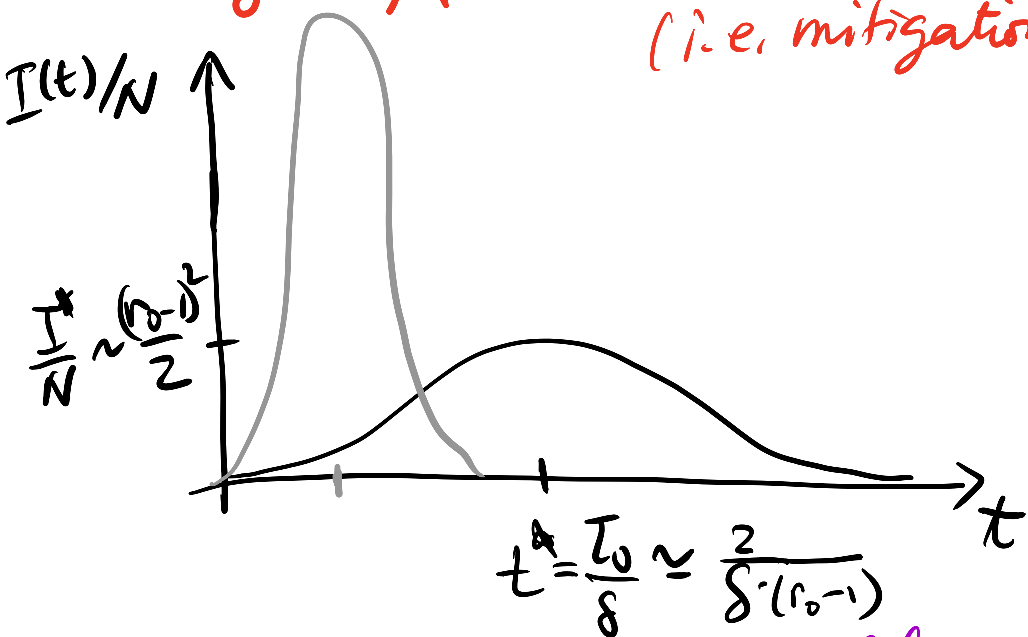


→ peak value = $\frac{I^*}{N} = \frac{1}{2}\left(1-\frac{1}{r_0}\right)^2$ ✓

→ occurs at time $t^* = \tau_0/\delta \approx \frac{2}{\delta(r_0-1)}$

for $r_0-1 \equiv x \ll 1$. $\tau_0 = \frac{2}{x^2}\tanh^{-1}x = \frac{2}{x}$

⇒ by reducing r_0, peak time shifted to later time (i.e. mitigation)



* Noted defficiencies of the SIR model:

- latency period: $S \to E \to I \to R$
  exposed
- age structure: $r(a)$
- asymptomatic infection: heterogeneity in δ.
- Spatial effect