C2. excitable system + relaxational oscillators

a) general consideration of 2d dynamical systems

$$\begin{cases} \dot{u} = f(u,v) \\ \dot{v} = g(u,v) \end{cases} \xrightarrow[v=v^*+\delta v]{u=u^*+\delta u} \begin{pmatrix} \delta\dot{u} \\ \delta\dot{v} \end{pmatrix} = M \begin{pmatrix} \delta u \\ \delta v \end{pmatrix}$$

Community matrix M:

$$M = \begin{pmatrix} f_u & f_v \\ g_u & g_v \end{pmatrix};$$

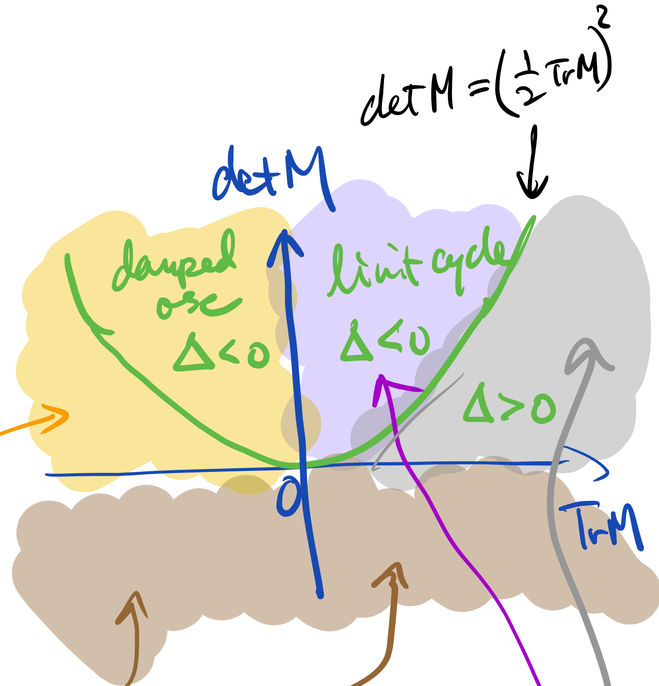
$$\det(M - \lambda I) = 0$$

$$\rightarrow (f_u - \lambda)(g_v - \lambda) - f_v g_u = 0$$

$$\lambda^2 - \lambda \underbrace{(f_u + g_v)}_{\mathrm{Tr}M} + \underbrace{f_u g_v - f_v g_u}_{\det M} = 0$$

(Note derivatives evaluated at u^*, v^*)

$$\lambda = \frac{1}{2}\mathrm{Tr}M \pm \sqrt{\underbrace{\left(\tfrac{1}{2}\mathrm{Tr}M\right)^2 - \det M}_{\Delta}}$$



→ Condition for stability:

$$\left.\begin{array}{l} \mathrm{Tr}M < 0 \\ \det M > 0 \end{array}\right\} \quad \lambda_\pm < 0$$

→ bistability (saddle pt)

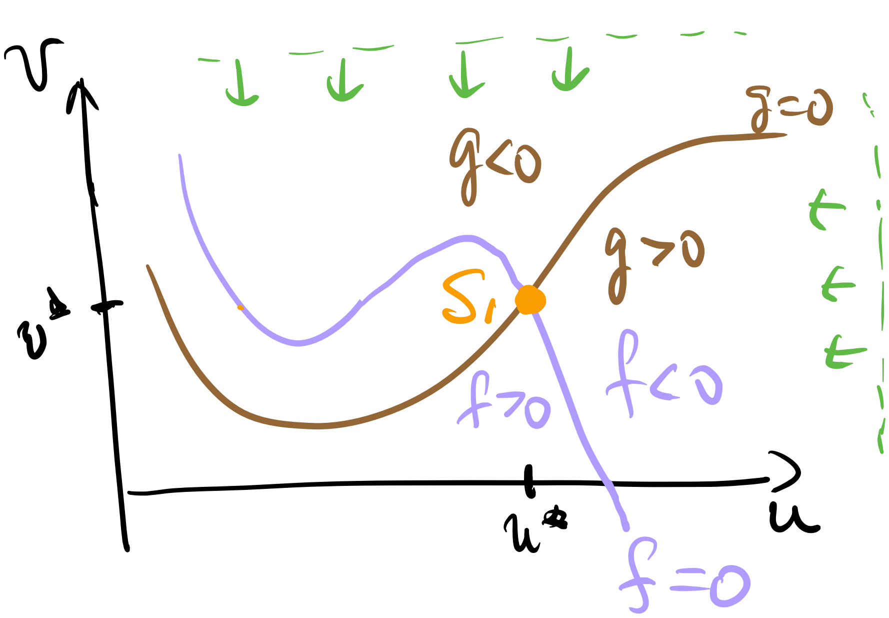
$\det M < 0$: $\lambda_+ > 0,\ \lambda_- < 0$

→ unstable spiral: $\mathrm{Tr}M > 0$, $\det M > \left(\frac{1}{2}\mathrm{Tr}M\right)^2$

→ unstable node:

$$\left.\begin{array}{l} \mathrm{Tr}M > 0 \\ \det M < \left(\frac{1}{2}\mathrm{Tr}M\right)^2 \end{array}\right\} \quad \lambda_\pm > 0$$

Now consider the following null cline structure



At fixed pt S_1

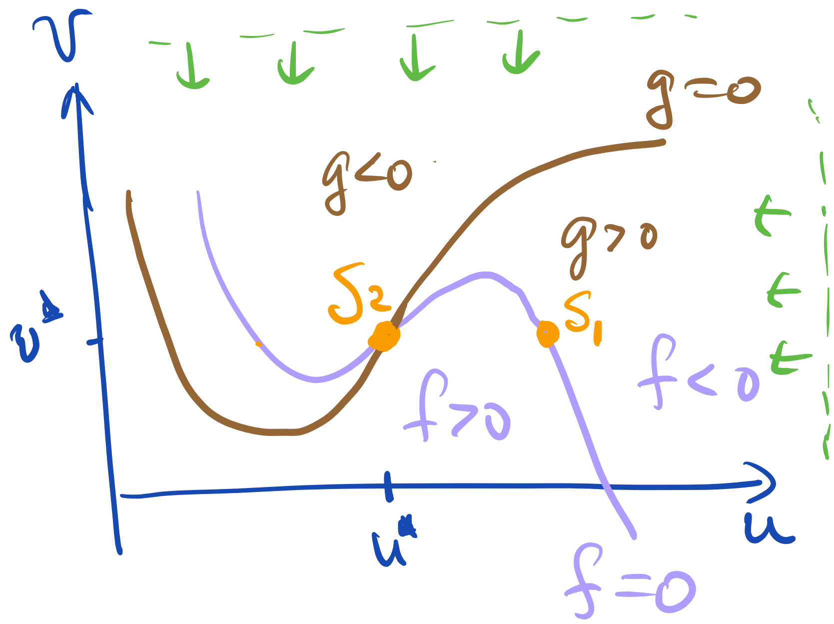
$f_u < 0, \; f_v < 0.$

$g_u > 0, \; g_v < 0$

$\rightarrow \operatorname{Tr} M < 0$

$\det M > 0$

Stable node or spiral



At fixed point S_2

$f_u > 0 \quad f_v < 0$

$g_u > 0 \quad g_v < 0$

$\operatorname{Tr} M \gtrless 0$?

$\det M \gtrless 0$?

$\rightarrow$ use topology of the null clines

$$\left.\frac{dv}{du}\right|_{g=0} = -\frac{g_u}{g_v} > \left.\frac{dv}{du}\right|_{f=0} = -\frac{f_u}{f_v} > 0$$

$dg = g_u du + g_v dv = 0$

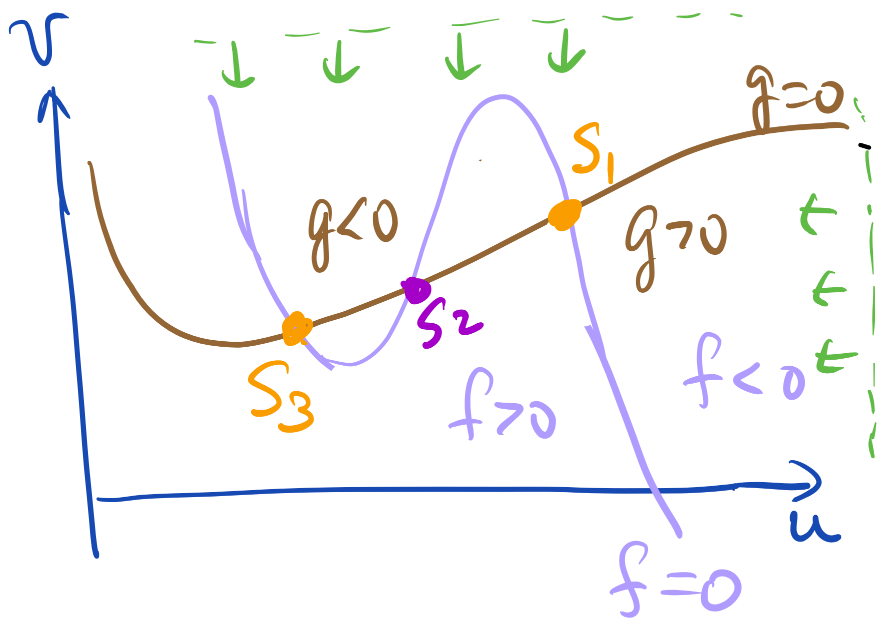
$$\frac{g_u}{g_v} < \frac{f_u}{f_v} \xrightarrow{f_v g_v > 0} f_u g_v > f_v g_u \rightarrow \det M > 0$$

$\rightarrow$ can admit osc soln if $0 < \operatorname{Tr} M < 2\sqrt{\det M}$

Another scenario:

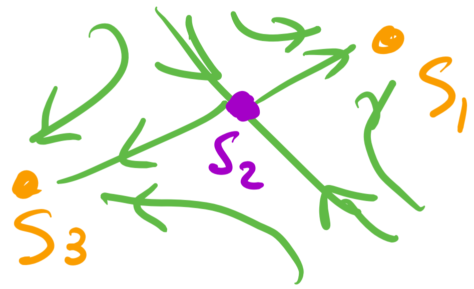
S_1 & S_3 both stable

S_2: $\left.\frac{dv}{du}\right|_{f=0} > \left.\frac{dv}{du}\right|_{g=0}$

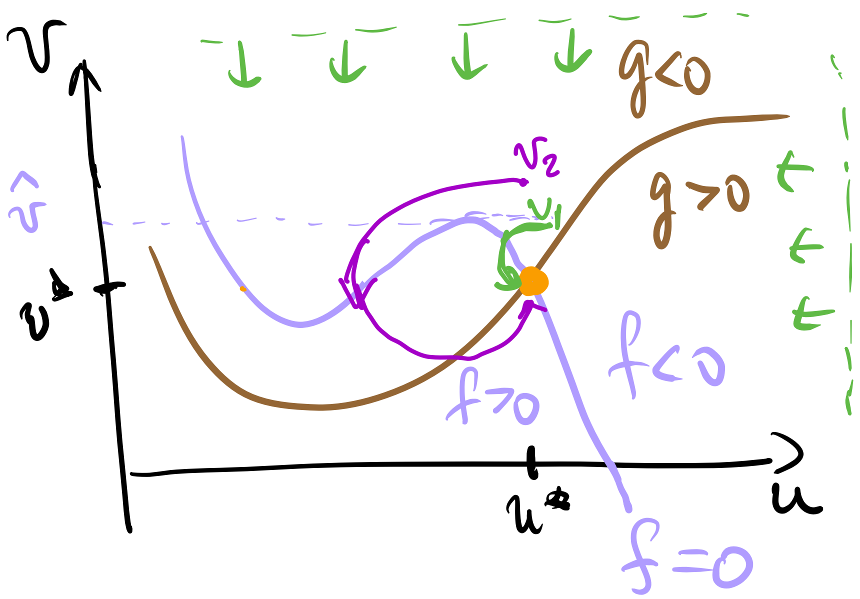


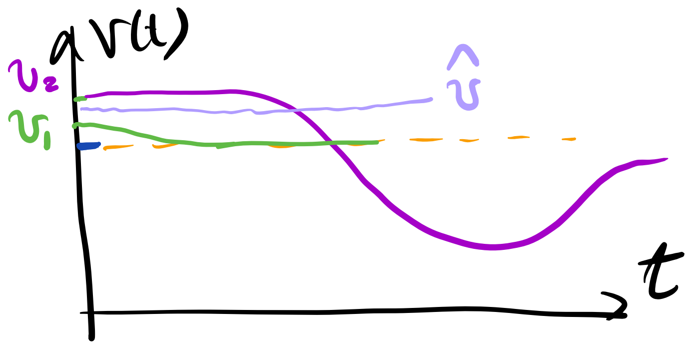
→ det M < 0

Saddle point (bistability)



✗ Effect of saddle or unstable spiral already manifested in stable phase as response to perturbation

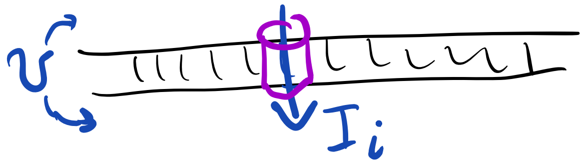




⇒ threshold phenomenon (excitable system)

b) FitzHugh-Nagumo (FHN) model

- widely used phenomenological model to capture threshold phenomena
- a simplified version of Hodgkin-Huxley model of neuron membrane potential dynamics



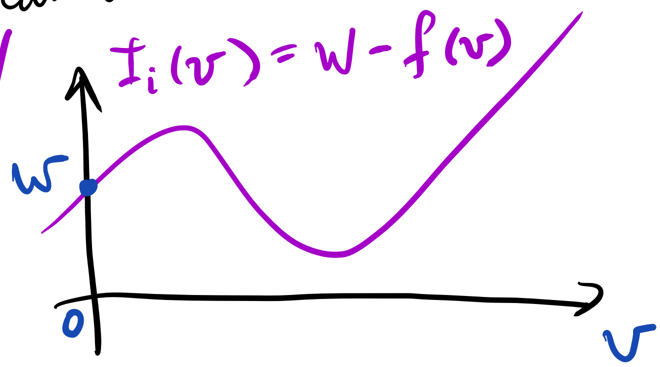
$$I(t) = C\frac{dv}{dt} + \underbrace{I_i(v)}_{\text{current thru ion channel}}$$

displacement current

$$\frac{1}{C}\frac{dv}{dt} = -I_i(v) + I_a$$

I_a: total current or app. current

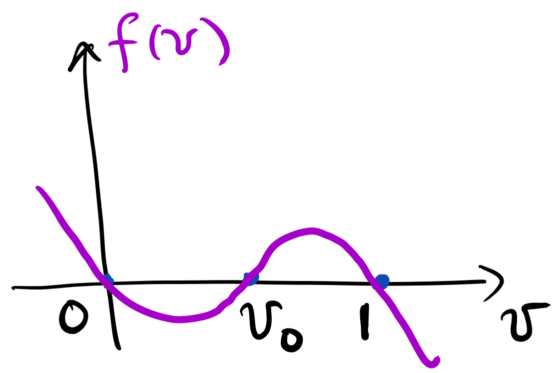
FHN



or

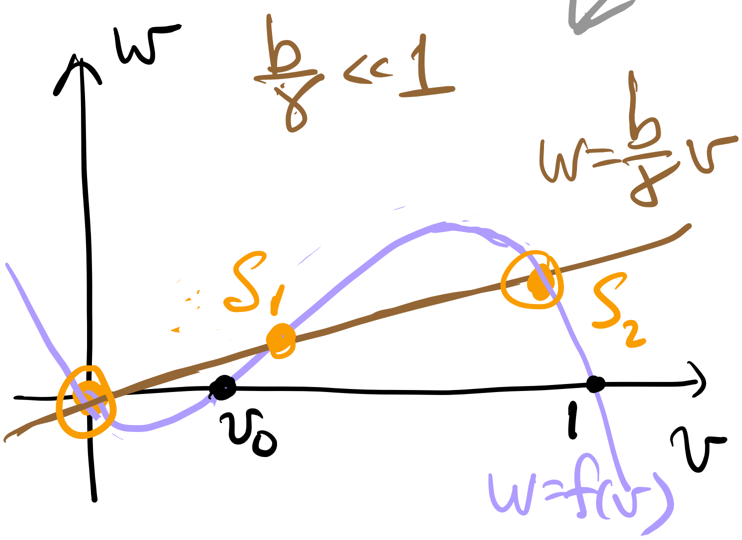
$$\begin{cases} \dfrac{dv}{dt} = f(v) - w + I_a \\ \dfrac{dw}{dt} = \underbrace{bv}_{\text{channel activation}} - \underbrace{\gamma w}_{\text{channel reversion}}, \end{cases} \quad (b, \gamma: \text{positive constants})$$

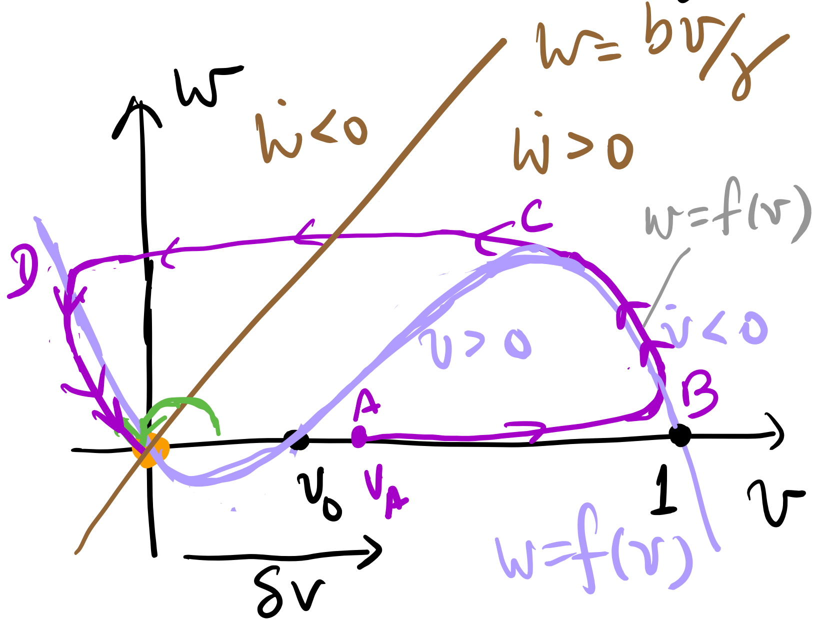
$$f(v) = v(v_0 - v)(v-1), \qquad 0 < v_0 < 1$$



Simplest phenomenological form constructed to capture neuron spikes and dynamics of excitable systems

Consider $I_a = 0$: admits bistable sol'n for small b/γ





Even if system is nominally stable (with (0,0) only f.p.)
it is excitable if $\delta v > v_0$ → v_0 = threshold.

⇒ Dynamics is particularly simple in the limit
$b \ll 1$, $b/\gamma \sim O(1)$.

$$\begin{cases} \dot{v} = f(v) - w \\ \dot{w} = b\left(v - \frac{\gamma}{b} w\right) \end{cases} \qquad \left|\frac{dw}{dt}\right| \ll \left|\frac{dv}{dt}\right|$$

(toilet-flushing mechanism)

A → B: rapid, with little change in w.

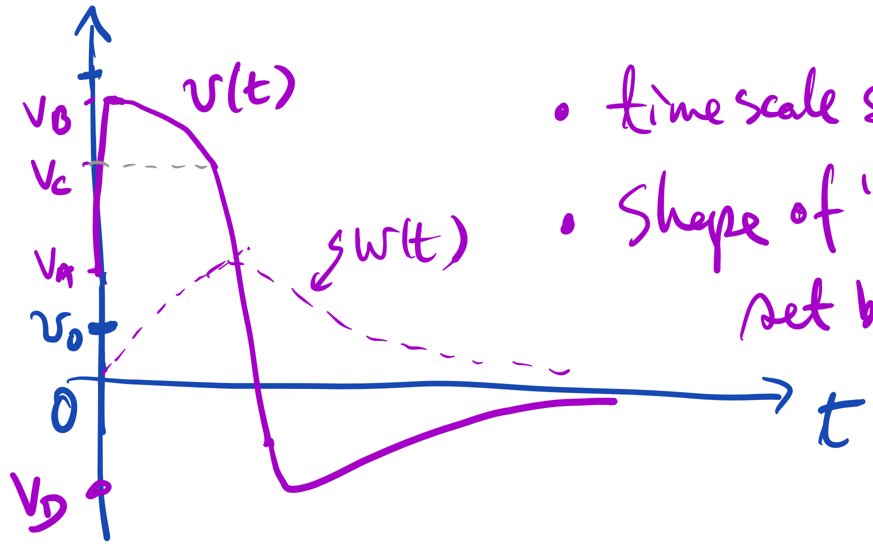
B → C: slow, dictated by time for w to reach w_c

→ can set $\frac{dv}{dt} = 0$ to get $w = f(v)$.

then solve for $\frac{dw}{dt} = b\left(f^{-1}(w) - \frac{\gamma}{b} w\right)$

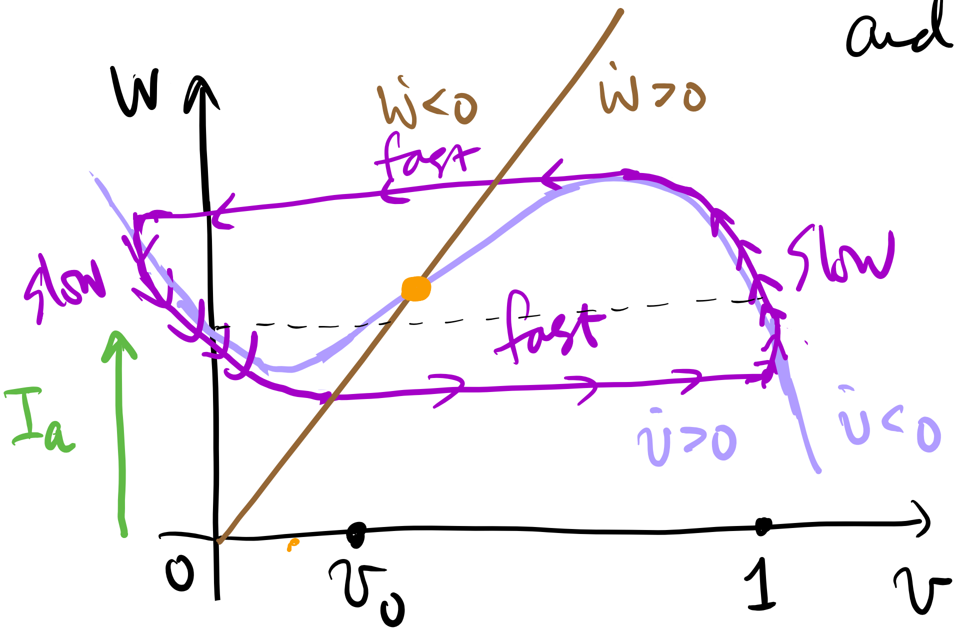
C → D: rapid; little change in w

D → 0: slow; again $w = f(v)$ and solve $\frac{dw}{dt}$.



- time scale set by dynamics of $w(t)$
- Shape of "action potential" set by form of $f(v)$

Sustained oscillation set by range of $I_a > 0$ and by the slope b/γ.



$$\begin{cases} \dot{v} = f(v) - w + I_a \\ \dot{w} = b\left(v - \gamma/b\, w\right) \end{cases}$$

$\Rightarrow$ relaxational oscillator (HW)

can compute (& control) period of oscillation

In Part II, we will examine the effect of spatial coupling and discuss the phenomenon of wave propagation in excitable medium.

Analogy to "ferromagnet"

$$F(m,h) = -m^2 + m^4 - h \cdot m$$