Part I. Population Dynamics and ecology

\# individuals of species i in a population : N_i

density : $\rho_i = N_i/V$.

→ this course: ignore discrete nature of N_i:
treat ρ_i as a continuous variable

[effect of demographic noise will be added
in a future edition of this course;
important for evolution dynamics
& certain ecological processes, e.g. invasion]

A. Intro to pop dynamics

1. Logistic model of pop. growth

- individuals replicate at rate r; no death

$$\frac{d\rho}{dt} = r\rho; \qquad \rho(t) = \rho_0 e^{rt} \to \infty$$

- carrying capacity $\tilde{\rho}$ (Common notation: K)

$$\frac{d\rho}{dt} = r\rho \cdot (1 - \rho/\tilde{\rho}) \quad \text{- Logistic eqn}$$

→ Simplest eqn to produce the phenomenology:

$$\frac{d\rho}{dt} \to 0 \quad \text{as } \rho \to \tilde{\rho}$$

a phenomenological description of the effect of starvation / crowding

* Exact soln:

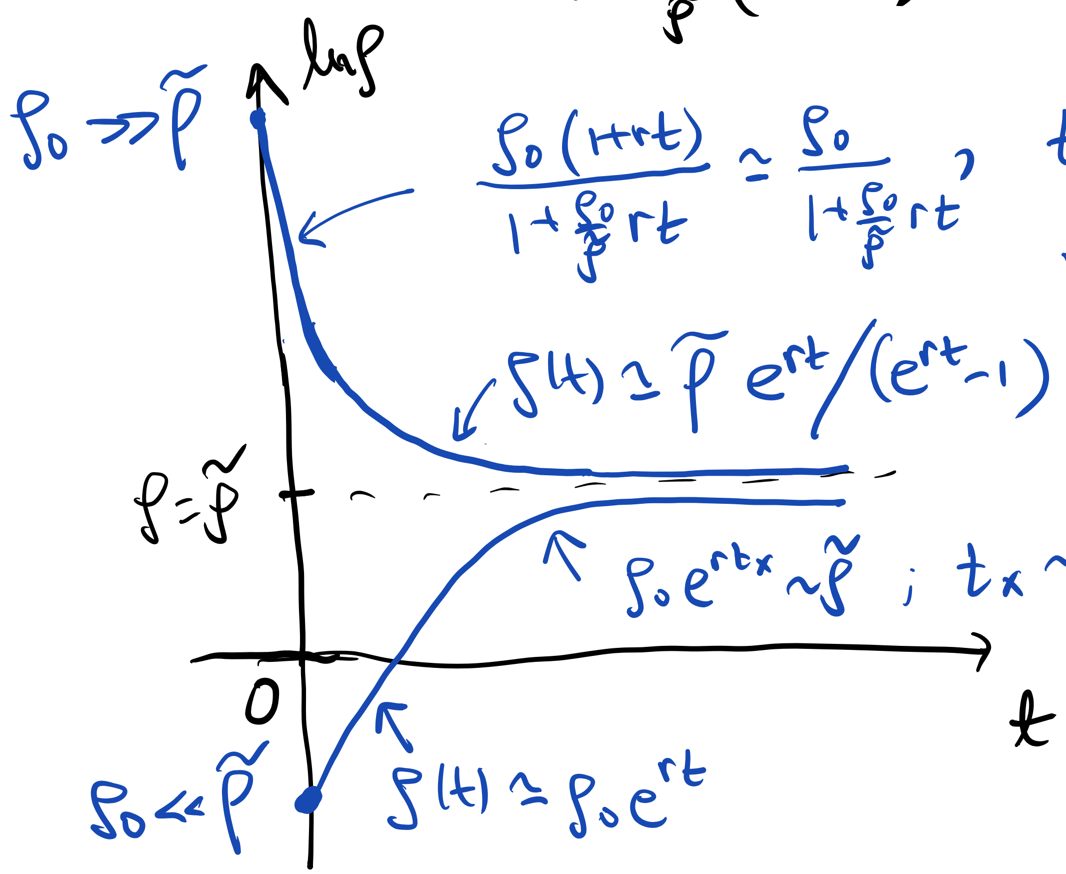
Init cond: $\rho(t=0) = \rho_0$

$$r\,dt = \frac{d\rho}{\rho(1-\rho/\tilde\rho)}$$

$$rt = \int_{\rho_0}^{\rho(t)} d\rho \left[\frac{1}{\rho} + \frac{1/\tilde\rho}{1-\rho/\tilde\rho} \right]$$

$$= \Big[\ln\rho - \ln(1-\rho/\tilde\rho) \Big]_{\rho_0}^{\rho(t)} = \ln\left(\frac{\rho}{1-\rho/\tilde\rho}\right)\Bigg|_{\rho_0}^{\rho(t)}$$

$$\to \rho(t) = \frac{\rho_0 e^{rt}}{1+\frac{\rho_0}{\tilde\rho}(e^{rt}-1)} = \begin{cases} \rho_0 & t=0 \\ \tilde\rho & t=\infty \end{cases}$$



$$\frac{\rho_0(1+rt)}{1+\frac{\rho_0}{\tilde\rho}rt} \simeq \frac{\rho_0}{1+\frac{\rho_0}{\tilde\rho}rt}, \quad t_x \sim \frac{\tilde\rho}{\rho_0}\frac{1}{r} \ll \frac{1}{r}.$$

to drop to $\rho_0/2$

$\rho(t) \simeq \tilde\rho\, e^{rt}/(e^{rt}-1)$ for $t > t_x$

$\rho_0 e^{rt_x} \sim \tilde\rho \; ; \; t_x \sim \frac{1}{r}\ln\tilde\rho/\rho_0$

$\rho(t) \simeq \rho_0 e^{rt}$

* Approach to stable fixed point

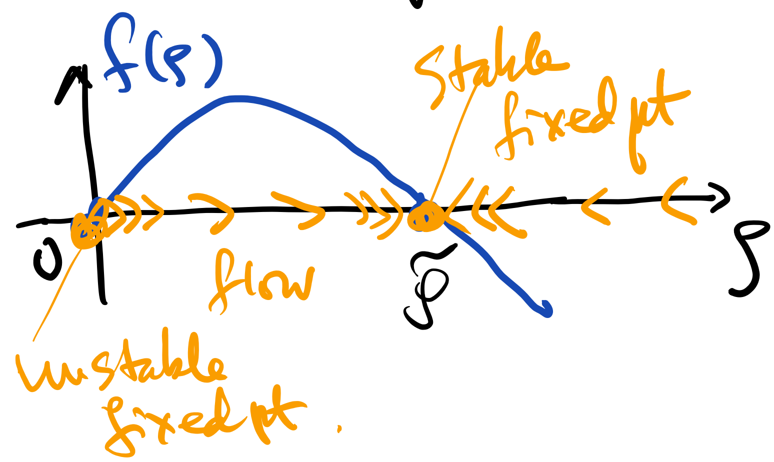
let $\rho(t) = \rho^* + \delta\rho(t)$; ρ^* is notation for f.p. $= \tilde{\rho}$

$$\frac{d}{dt}\delta\rho = r(\tilde{\rho} + \delta\rho)\left(1 - (\tilde{\rho} + \delta\rho)/\tilde{\rho}\right) = -r\,\delta\rho$$

$\delta\rho(t) \propto e^{-rt}$ same time scale of approach from above + below.

* Soln from "visual inspection" of the ODE.

$$\frac{d\rho}{dt} = \underbrace{r\rho\left(1 - \rho/\tilde{\rho}\right)}_{f(\rho)}$$



Warning: this is a phenomenological model; be careful about mechanistic interpretation! (e.g. does not describe batch growth)

InSeeD, we will discuss the meaning of this eqn for microbial growth in the context of nutrient uptake + metabolism

2. Balance of replication + predation

Include the effect of pop loss into logistic growth

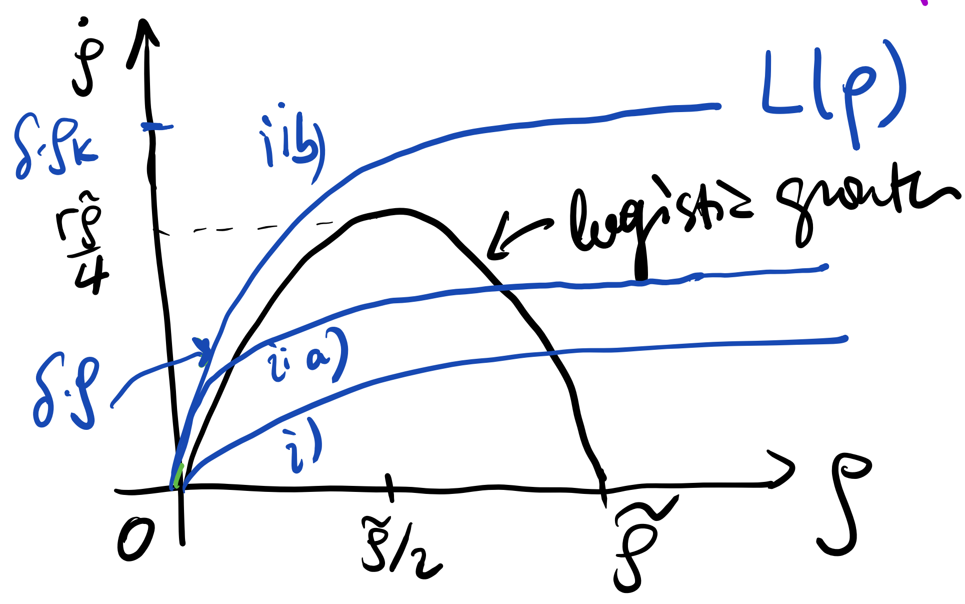
$$\frac{d\rho}{dt} = r\rho\left(1-\frac{\rho}{\tilde{\rho}}\right) - L(\rho)$$

- constant death rate : shifts reprod. rate r.
- effect of predation generally density-dependent
  e.g., killing of bacteria by phage or eukaryotes

$$L(\rho) = \frac{\delta \rho}{1+\rho/\rho_K}$$

δ : max loss rate

(note : effect of bacteria on predator ignored here)

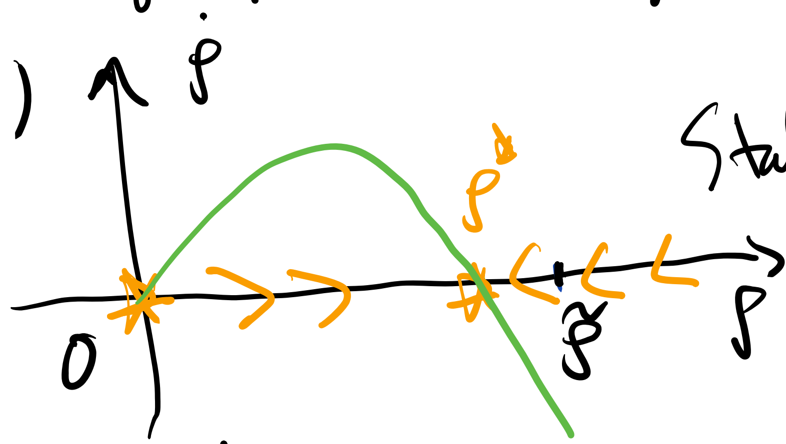


Case i) $\delta < r$ $\qquad \rho^* > 0$

ii) $\delta > r$ <
- ii a) $\delta \cdot \rho_K \lesssim r\tilde{\rho}/4$ $\qquad \rho^* = ?$
- ii b) $\delta \cdot \rho_K \gtrsim r\tilde{\rho}/4$ $\qquad \rho^* = 0$

a) graphical analysis

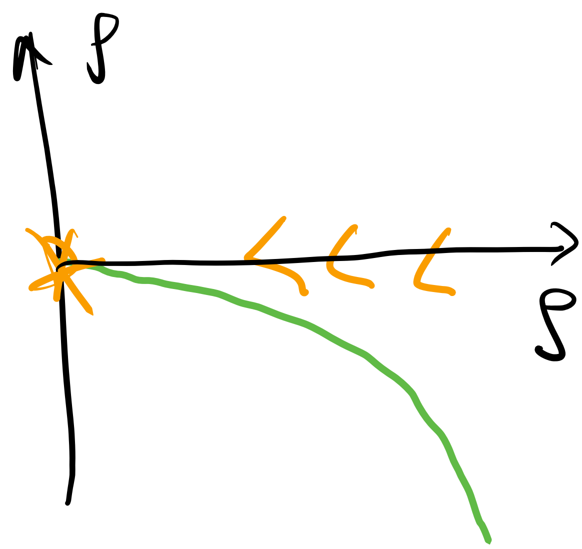
i)



Stable fixed pt at $\rho^* \lesssim \tilde\rho$

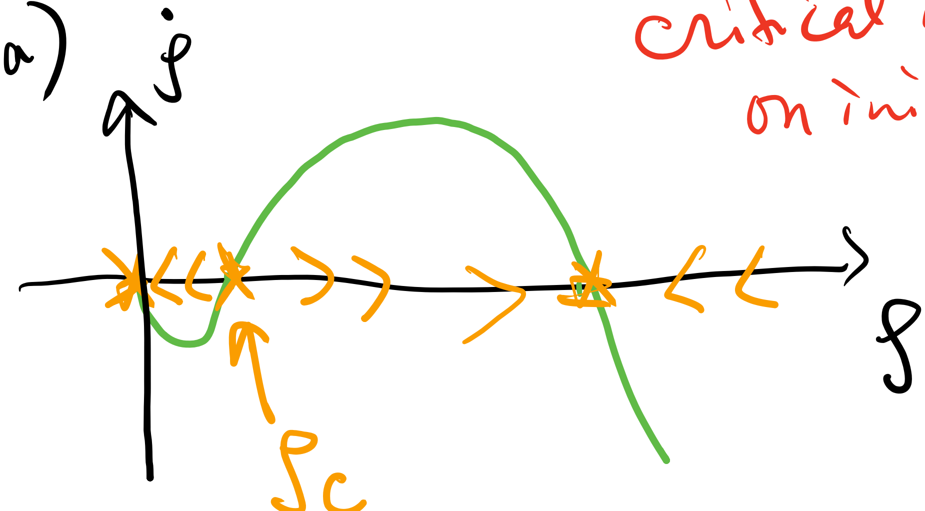
carrying capacity moderately reduced.

iib)



Pop always driven to extinction

ii a)



critical dependence on init density ρ_0

"Allee Effect"

two phases:

if $\rho_0 < \rho_c$, then $\rho(t\to\infty) \to 0$ (extinction)

if $\rho_0 > \rho_c$, then $\rho(t\to\infty) \to \rho^* \lesssim \tilde\rho$ (Stable existence)

→ life-or-death depends on init condition.

b) Analytical Study : quantitative dependence of ρ^* on system parameters

$$\frac{d\rho}{dt} = r\rho\left(1-\frac{\rho}{\tilde{\rho}}\right) - \frac{\delta \cdot \rho}{1+\rho/\rho_K}$$

→ make dimensionless :

$$u = \frac{\rho}{\tilde{\rho}}, \quad rt = \tau, \quad \frac{\delta}{r} \equiv \alpha, \quad \frac{\rho_K}{\tilde{\rho}} \equiv \kappa$$

$$\frac{du}{d\tau} = u(1-u) - \frac{\alpha u}{1+u/\kappa}$$

two dimensionless parameters :

$\alpha = \frac{\delta}{r}$ large α = large predation

$\kappa = \rho_K/\tilde{\rho}$ large κ = large predation

Steady-state :

$$\frac{du}{d\tau} = 0 \rightarrow \underbrace{u^* = 0}_{\text{extinction}} \quad \underline{\underline{\text{or}}} \quad \underbrace{1 - u^* = \frac{\alpha}{1+u^*/\kappa}}_{\text{existence?}}$$

$$(u^*)^2 + (\kappa-1)u^* + (\alpha-1)\kappa = 0$$

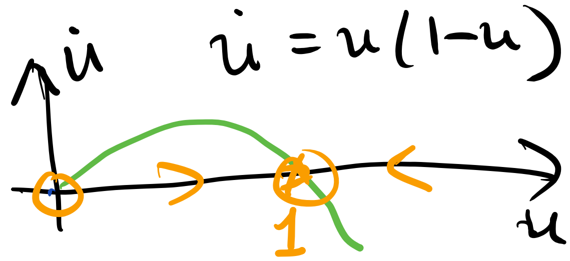
$$u^* = \frac{1-\kappa}{2} \pm \sqrt{\left(\frac{\kappa-1}{2}\right)^2 - (\alpha-1)\kappa}$$

existence requires soln with $u^* > 0$

$\underline{\alpha = 0}$. $u^* = \frac{1-\kappa}{2} \pm \sqrt{\left(\frac{\kappa+1}{2}\right)^2} = \frac{1-\kappa}{2} \pm \frac{1+\kappa}{2}$

$$= \begin{cases} 1 & \rightarrow \rho^* = \tilde{\rho} \\ -\cancel{\kappa} & \rightarrow \text{unbiological} \end{cases}$$

$\dot{u} = u(1-u)$



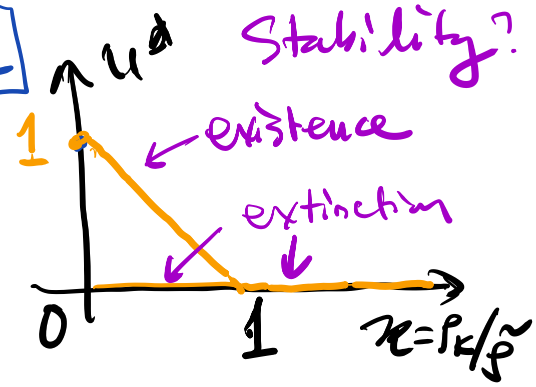
$\alpha = 1$

borderline case with $\delta = r$

$$u^* = \frac{1-\kappa}{2} \pm \left| \frac{\kappa - 1}{2} \right|$$

$$= \begin{cases} 0, \ \cancel{1-\kappa} & (\kappa > 1) \\ 1-\kappa, \ 0 & (\kappa < 1) \end{cases}$$

$\boxed{\alpha = 1}$



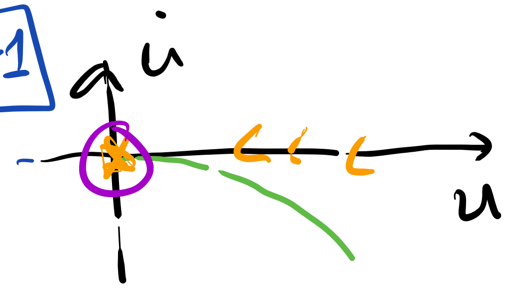
$$\dot{u} = u(1-u) - \frac{u}{1+u/\kappa}$$

$$= \frac{u}{1+u/\kappa} \cdot \left[(1-u)(1+u/\kappa) - 1 \right]$$

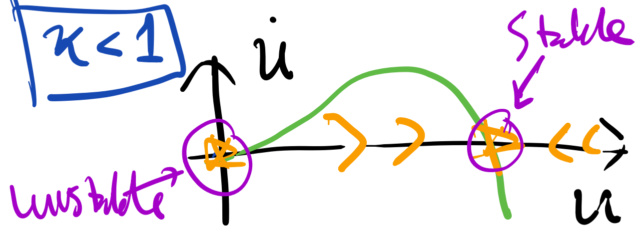
$$= \frac{u}{1+u/\kappa} \left[\frac{u}{\kappa}(1-u) - u \right]$$

$$= \frac{u^2}{u+\kappa} \left[1-\kappa-u \right]$$

$\boxed{\kappa > 1}$



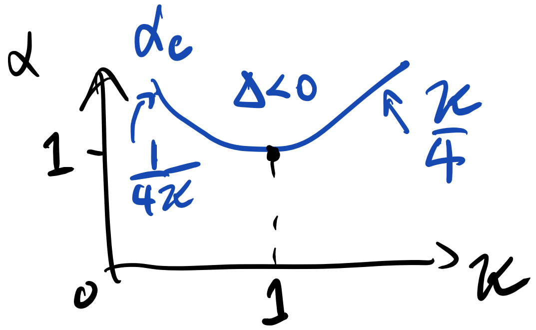
$\boxed{\kappa < 1}$



c) phase diagram : which phase is exhibited for what parameters (α, κ)

$$u^* = \frac{1-\kappa}{2} \pm \sqrt{\underbrace{\left(\frac{\kappa-1}{2}\right)^2 - (\alpha-1)\kappa}_{\Delta \ \text{(discriminant)}}} > 0$$

* $\Delta = 0 \quad \rightarrow \quad \kappa^2 + 2\kappa + 1 = 4\alpha_c \kappa$



$$4\alpha_c = \kappa + 2 + \frac{1}{\kappa}.$$

$$4\alpha_c' = 1 - \frac{1}{\kappa^2} = 0$$

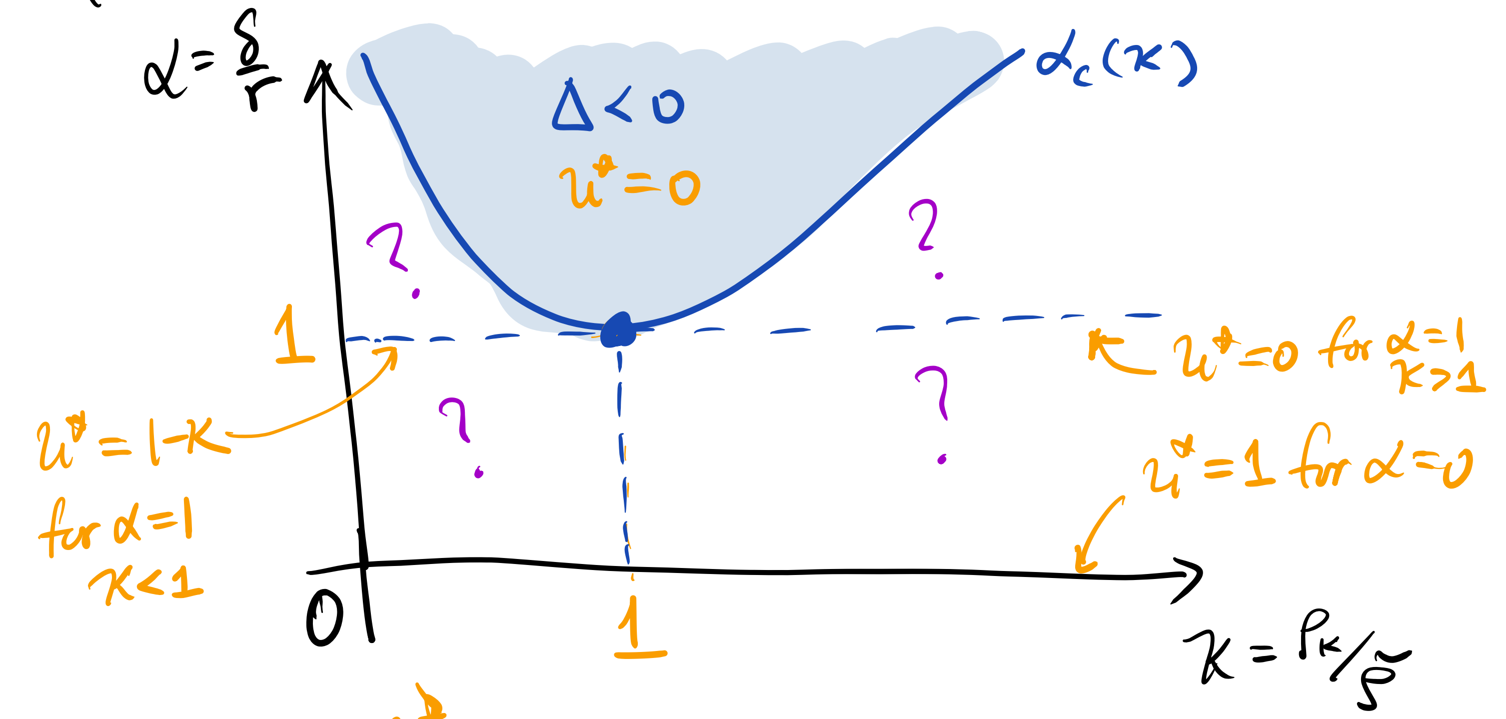
min at $\kappa = 1$, $\alpha_c = 1$ $\left(4\alpha_c'' = \frac{2}{\kappa^3} > 0\right)$.

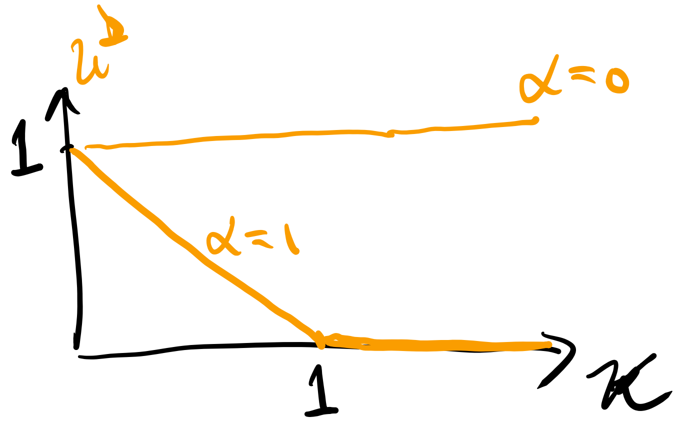
* for $\alpha > \alpha_c$, $\Delta < 0$ → only sol'n is $u^* = 0$ (extinction !)

no real soln

* for $\alpha < \alpha_c$; $\Delta < 0$ → one or two sol'n possible $u^* > 0$?

phase diagram:

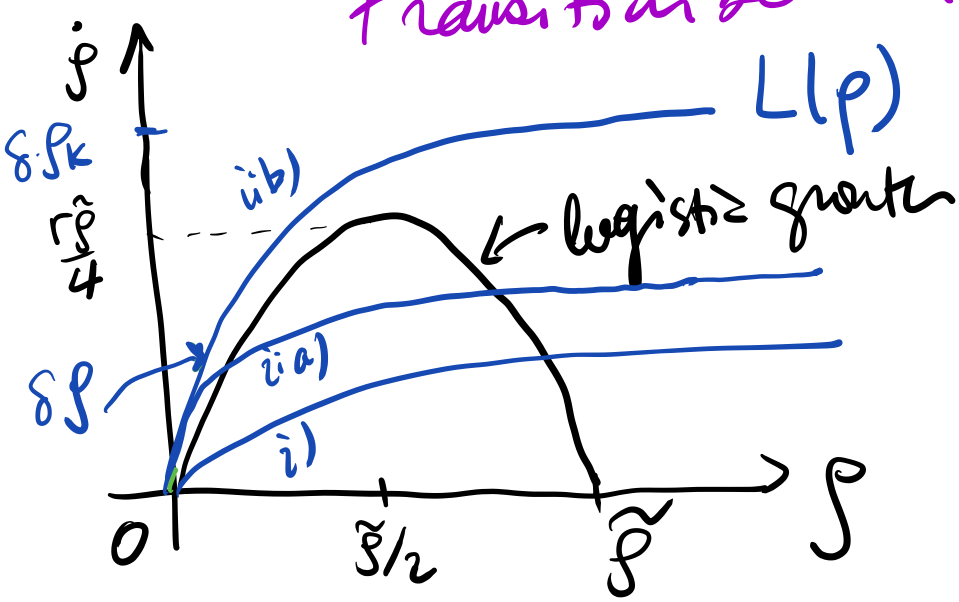




$u^*(\alpha, \kappa)$ for other values of α ?

Recall 3 phases discussed at begining

transition between phases = "bifurcation"



i) $\delta < r$ $(\alpha < 1)$

ii) $\delta > r$ $(\alpha > 1)$

ii a) $\delta \cdot \rho_K \lesssim r\tilde{\rho}/4$ $\left(\alpha < \kappa/4\right)$

ii b) $\delta \cdot \rho_K \gtrsim r\tilde{\rho}/4$ $\left(\alpha > \frac{\kappa}{4}\right)$