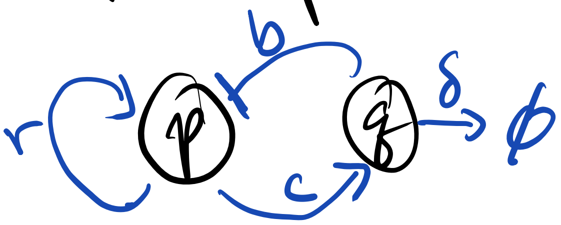
A3. Simple models of two-species interaction

a) Predator-Prey system

- two species:



prey (density p)
predator (density q)

$$\frac{dp}{dt} = rp - bpq$$
$$\frac{dq}{dt} = cpq - \delta q$$

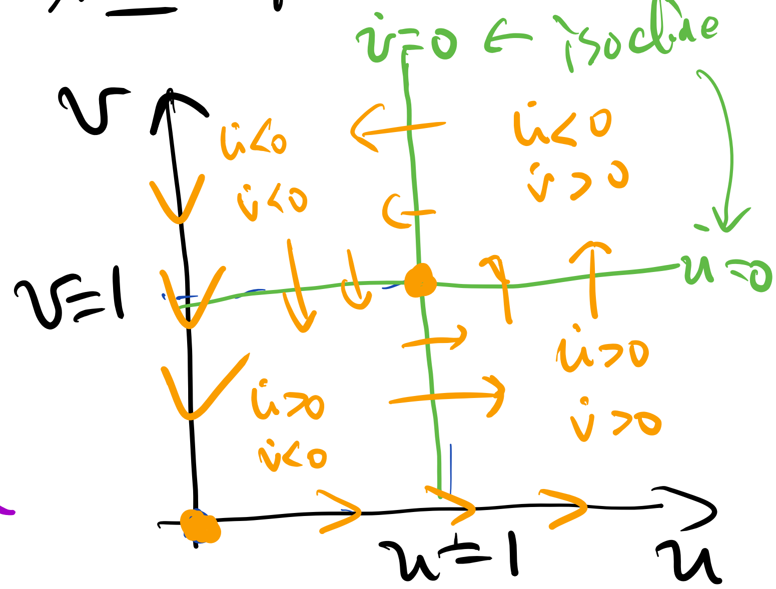
Lotka-Volterra Model

Volterra (1926):
[Lotka (1920) introduced same eqn to describe chemical reactions]

make dimensionless: $u = p/\frac{\delta}{c}$, $v = q/\frac{r}{b}$, $\tau = r\cdot t$, $\alpha = \delta/r$

$$\begin{cases} \frac{du}{d\tau} = u(1-v) \\ \frac{dv}{d\tau} = \alpha v(u-1) \end{cases}$$

Sketch of 2d flow



fixed pt: $(\dot{u}, \dot{v}) = (0,0), (1,1)$

→ dynamics around (1,1):
damped or unstable spiral

let $\begin{cases} u = 1+x \\ v = 1+y \end{cases}$ → $\begin{cases} \frac{dx}{d\tau} = -y \\ \frac{dy}{d\tau} = \alpha x \end{cases}$ or $\frac{d}{d\tau}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ \alpha & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

Solution:

$$\begin{pmatrix} x(\tau) \\ y(\tau) \end{pmatrix} = \begin{pmatrix} x_0 e^{\lambda\tau} \\ y_0 e^{\lambda\tau} \end{pmatrix}, \text{ or } \frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ \alpha & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\Rightarrow \det\left[\begin{pmatrix} 0 & -1 \\ \alpha & 0 \end{pmatrix} - \lambda I\right] = 0. \quad \begin{vmatrix} -\lambda & -1 \\ \alpha & -\lambda \end{vmatrix} = \lambda^2 + \alpha = 0$$

$$\boxed{\lambda = \pm i\sqrt{\alpha}}$$

$$\Rightarrow \begin{cases} x(\tau) = x_0 e^{\pm i\sqrt{\alpha}\tau} \\ y(\tau) = y_0 e^{\pm i\sqrt{\alpha}\tau} \end{cases}$$ soln depends on init cond.

full solution: $\frac{du/d\tau}{dv/d\tau} = \frac{du}{dv} = \frac{u(1-v)}{\alpha v(u-1)}$

$$\alpha \frac{du}{u}(u-1) = \frac{dv}{v}(1-v)$$

$$\alpha u - \ln u^\alpha = \ln v - v + H$$ ← const.

$$\boxed{H(u,v) = \alpha u + v - \alpha \ln u - \ln v}$$

– conserved quantity!

at $u=1, v=1$, $H(1,1) = 1+\alpha \equiv H_0$

→ $H(u,v)$ has global min at $u=1, v=1$

$$\frac{\partial H}{\partial u} = \alpha - \frac{\alpha}{u} = 0 \text{ at } u^* = 1; \quad \frac{\partial^2 H}{\partial u^2} = \frac{\alpha}{u^2} > 0$$

$$\frac{\partial H}{\partial v} = 1 - \frac{1}{v} = 0 \text{ at } v^* = 1; \quad \frac{\partial^2 H}{\partial v^2} = \frac{1}{v^2} > 0$$

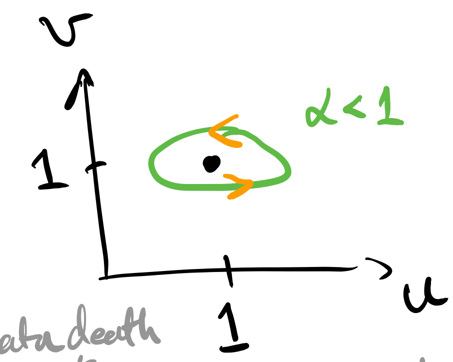
trajectory of orbits:

- $H \gtrsim H_0$: let $u = 1+x$, $v = 1+y$.

$$\alpha + \alpha x + 1 + y - \ln(1+x)^\alpha - \ln(1+y) = H$$

$$(\cancel{\alpha x} + \cancel{y}) - \alpha(\cancel{x} - \tfrac{x^2}{2} \cdots) - (\cancel{y} - \tfrac{y^2}{2} \cdots) = H - H_0$$

$$\rightarrow \alpha \frac{x^2}{2} + \frac{y^2}{2} = H - H_0 \quad \text{(ellipse)}$$



$y = 0 \qquad x = \pm\sqrt{\frac{2}{\alpha}(H - H_0)}$

long-axis along x if $\alpha < 1$ or $\delta < r$

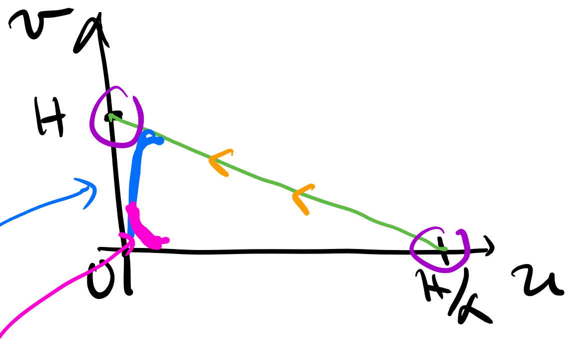
predator death rate

prey growth rate

- $H \gg H_0$: look at $\alpha u + v - \alpha \ln u - \ln v = H$.

for large u, v, can neglect $\ln u$, $\ln v$.

$\rightarrow \alpha u + v \simeq H$.



- breaks down when $u, v \to 0$

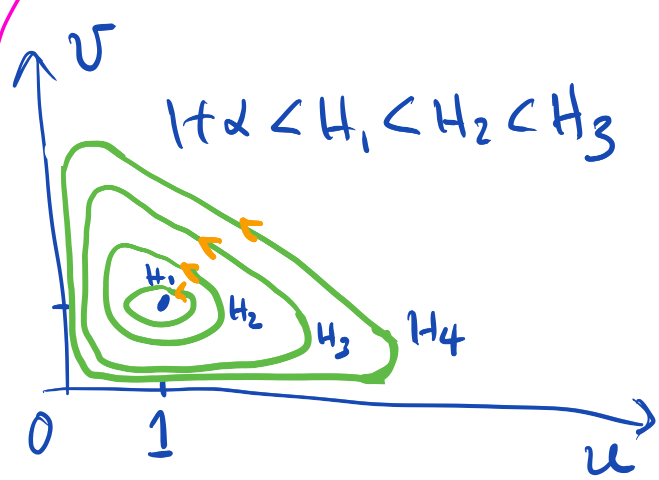
Consider $v \simeq H$, $u \to 0$.

here $v \simeq H + \alpha \ln u$

- for both $u, v \ll 1$

$-H = \alpha \ln u + \ln v = \ln v \cdot u^\alpha$

$v = u^{-\alpha} e^{-H}$



Overall lesson: conserved quantity → periodic orbit.

but solution completely dependant on init cond.

b) break conserved quantity

e.g. include logistic growth of prey.

$$\begin{cases} \frac{dp}{dt} = rp\left(1 - p/\tilde{p}\right) - bpq \\ \frac{dq}{dt} = cpq - \delta q \end{cases}$$

effect of prey's carrying capacity

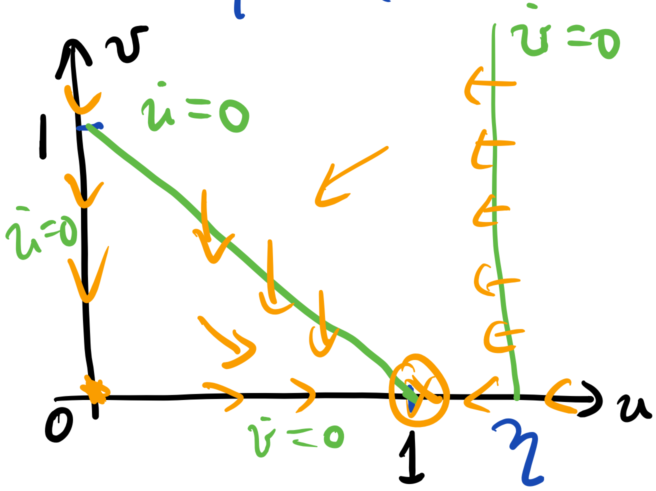
dimensionless: $u = p/\tilde{p}$, $v = q\frac{b}{r}$, $\tau = r \cdot t$, $\alpha = \delta/r$, $\eta = \frac{\delta/c}{\tilde{p}}$

$$\begin{cases} \frac{du}{d\tau} = u(1 - u - v) \\ \frac{dv}{d\tau} = \alpha v \cdot \left(\frac{u}{\eta} - 1\right) \end{cases}$$

$\dot{u} = 0$; $u = 0$ or $u + v = 1$

$\dot{v} = 0$: $v = 0$ or $u = \eta$

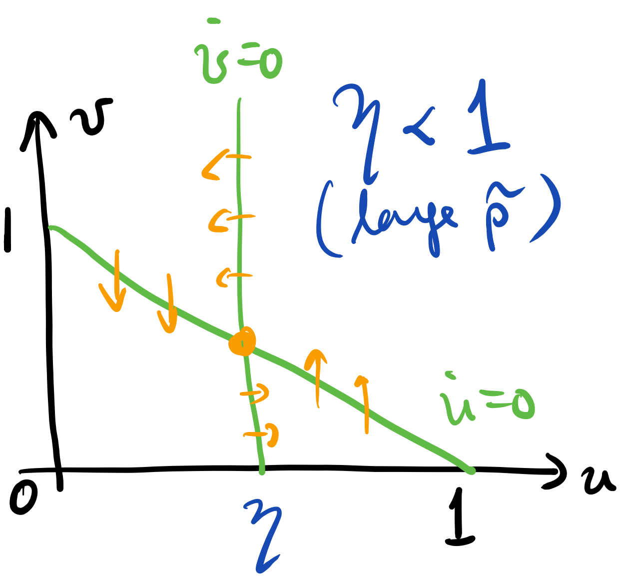
$\eta > 1$ (or small $\tilde{p}$: small carrying capacity for prey)



predator extinct

prey → carrying capacity

⇒ small carrying capacity allows prey to evade predator



Check for stability and oscillation

$u^* = \eta$, $v^* = 1 - u^* = 1 - \eta$

$u = \eta + x$. $v = 1 - \eta + y$.

for small x, y

$$\begin{cases} \frac{dx}{d\tau} = -\eta \cdot (x + y) \\ \frac{dy}{d\tau} = \alpha(1-\eta) \cdot \frac{x}{\eta} \end{cases}$$

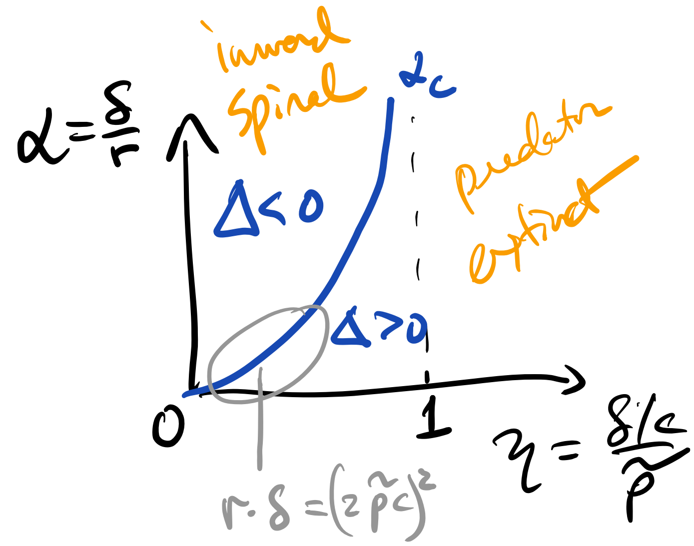
$$\begin{bmatrix} -\eta & -\eta \\ \frac{\alpha(1-\eta)}{\eta} & 0 \end{bmatrix} - \lambda I = 0 \quad \rightarrow \quad \det \begin{vmatrix} -\eta-\lambda & -\eta \\ \frac{\alpha(1-\eta)}{\eta} & -\lambda \end{vmatrix} = 0$$

$$\lambda^2 + \eta\lambda + \alpha(1-\eta) = 0.$$

$$\lambda = -\frac{\eta}{2} \pm \sqrt{\underbrace{\left(\frac{\eta}{2}\right)^2 - \alpha(1-\eta)}_{\Delta}}$$

$$\Delta = \left(\frac{\eta}{2}\right)^2 - \alpha(1-\eta) = 0$$

$$\rightarrow \quad \alpha_c = (\eta/2)^2/(1-\eta)$$



if $\Delta < 0$: $\lambda = -\frac{\eta}{2} \pm i\sqrt{|\Delta|}$ damped oscillation (towards coexistence)

if $\Delta > 0$: $\lambda = -\frac{\eta}{2} \pm \underbrace{\sqrt{\Delta}}_{< \frac{\eta}{2}} < 0$ Stable coexistence (Stable node)

$\Rightarrow$ Stable oscillation exhibited by the simple Lotka-Volterra model is not robust.

Note: if $Re\{\lambda\} > 0$ and $Im\{\lambda\} \neq 0$, and further if u and v are bounded, then obtain <u>Stable limit cycle</u> (Poincare-Bendixson Theorem)

$\rightarrow$ Will show this occurs when saturation of predation is included