5. Amplitude eqn and pattern selection

Turing: finite wavelength instability

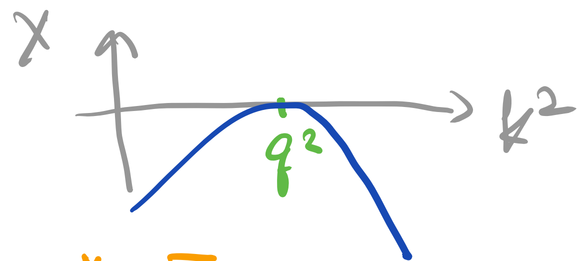
Q: What "happen" to the unstable modes?

A: analyze their dynamics using "amplitude eqn"

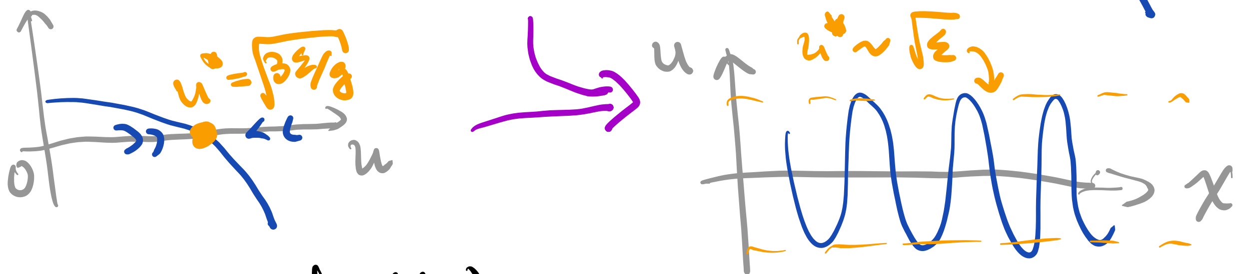
a). derivation of amplitude eqn:

To simplify the math, we consider an isotropic one-variable model with built-in finite wavelength instability

$$\frac{\partial u}{\partial t} = \underbrace{\varepsilon u - \frac{g}{3}u^3}_{u^* > 0} + \underbrace{X(\nabla^2)u}_{\text{finite-}k\text{ instability}}$$



$\varepsilon > 0,\ g > 0$:



Simplest form of $X(k)$:

$$X = -(q^2 - k^2)^2 = -(q^4 - 2q^2k^2 + k^4)$$

in real space, $X \cdot u = -(q^4 u + 2q^2 \nabla^2 u + \nabla^2 \nabla^2 u)$

$$= -(q^2 + \nabla^2)^2 u$$

$$\Rightarrow \frac{\partial u}{\partial t} = \left[\varepsilon - (q^2 + \nabla^2)^2\right]u - \frac{g}{3}u^3; \quad 0 < \varepsilon \ll 1; \quad g > 0$$

– Swift-Hohenberg model (1977)

(describes the dynamics of Rayleigh-Bénard instability which occurs for fluid heated from below)

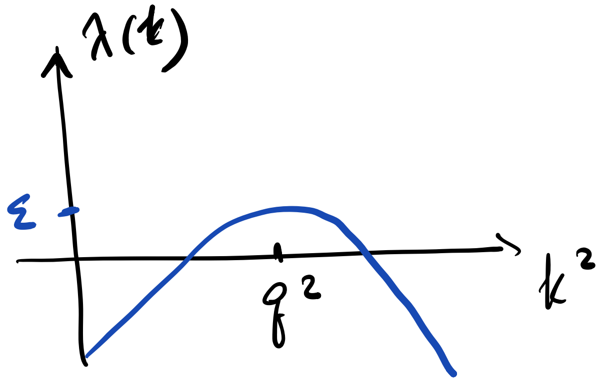
– include u^2 term (Haken model)
if $u \to -u$ symmetry absent

* linear stability

let $u = \delta u(t)\, e^{ikx}$

then $\partial_t \delta u = \left[\varepsilon - (q^2 - k^2)^2\right] \cdot \delta u$

$\to \lambda(k) = \varepsilon - (q^2 - k^2)^2$



→ dispersion mimics those of dynamical systems exhibiting Turing instability

→ model system for studying rules of pattern formation

* include nonlinearity:

Consider system in 2d, and linear instability results in stripes along $\hat{x}$-direction: $\vec{q} = q\hat{x}$

Set $\partial_t u = 0 \xrightarrow{SH} u_0(x,y) = \sum_n a \cos(nqx) + b \sin(nqx)$

$\Rightarrow$ is the soln $u_0(x,y)$ stable to perturbation?

allow variation in amplitude:

$u(\vec{r},t) = A(\vec{r},t) \cdot u_0(\vec{r})$

spatial variation with $k \ll q$

approx: $u_0 \simeq a \cdot \cos qx + b \cdot \sin qx$

(ignore higher-order harmonics)

write as: $u(\vec{r},t) = A(\vec{r},t)\, e^{iqx} + \underline{A^*(\vec{r},t)}\, e^{-iqx}$

Complex conjugate of A

(more convenient representation)

→ substitute $u(\vec{r},t)$ into Swift-Hohenberg eqn

$$\frac{\partial u}{\partial t} = \frac{\partial A}{\partial t} e^{iqx} + \frac{\partial A^*}{\partial t} e^{-iqx}$$

$$\nabla^2 u = \nabla^2 A\, e^{iqx} + \nabla^2 A^*\, e^{-iqx} - q^2 A e^{iqx} - q^2 A^* e^{-iqx}$$
$$+ 2iq\, \partial_x A\, e^{iqx} - 2iq\, \partial_x A^*\, e^{-iqx}$$

$$(q^2 + \nabla^2) u = (\nabla^2 A) e^{iqx} + (\nabla^2 A^*) e^{-iqx}$$
$$+ 2iq (\partial_x A) e^{iqx} - 2iq (\partial_x A^*) e^{-iqx}$$

$$\left[\varepsilon - (q^2 + \nabla^2)^2\right] u = e^{iqx} \left[\varepsilon - \left(2iq \frac{\partial}{\partial x} + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)^2\right] A$$
$$+ e^{-iqx} \left[\varepsilon - \left(-2iq \frac{\partial}{\partial x} + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)^2\right] A^*$$

$$\Rightarrow \quad \frac{\partial A}{\partial t} = \left[\varepsilon - \left(2iq \frac{\partial}{\partial x} + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)^2\right] A$$

+ contribution from u^3 term.

$$u^3 = \underbrace{\cancel{A^3 e^{3iqx}}}_{\text{higher-harmonic}} + 3|A|^2 A\, e^{iqx} + 3|A|^2 A^* e^{-iqx} + \underbrace{\cancel{A^{*3} e^{-3iqx}}}_{\text{higher-harmonics}}$$

$$\frac{\partial A}{\partial t} = \left[\varepsilon - \left(2iq\frac{\partial}{\partial x} + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)^2\right]A - g|A|^2 A$$

Amplitude eqn for SH system.

Stationary sol'n: $A(\vec{r}) = \hat{A}_{\vec{k}}\, e^{i\vec{k}\cdot\vec{r}}$ (note: $\hat{u}(\vec{k})$ peaked at $\vec{k}=\vec{q}$)

$$\rightarrow 0 = \left[\varepsilon - (2qk_x + k_x^2 + k_y^2)^2\right]\hat{A}_k - g|\hat{A}_{\vec{k}}|^2\hat{A}_{\vec{k}}.$$

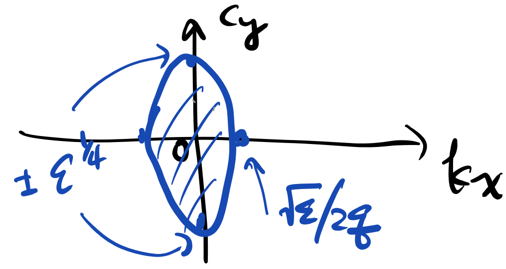
$$\rightarrow \quad |\hat{A}_{\vec{k}}| = \frac{1}{\sqrt{g}}\left[\varepsilon - (2qk_x + k_x^2 + k_y^2)^2\right]^{1/2}$$

for $0 < \varepsilon \ll 1$, $|\hat{A}_{k=0}| \simeq \sqrt{\varepsilon/g}$

$|\hat{A}_k| > 0$ also for small k_x, k_y

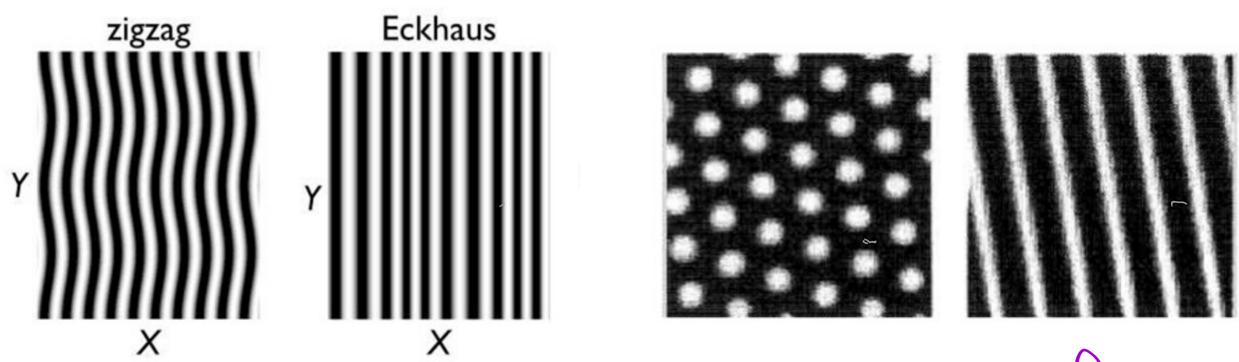
$$(2qk_x + k_x^2 + k_y^2)^2 < \varepsilon/g$$

or $(2qk_x + k_y^2)^2 < \varepsilon/g$.



$\Rightarrow$ assume stripe, get stripe and more!

Q1: are stripes stable to low-k perturbations?



Q2: are stripes stable to squares or hexagons?

b). Stability of the stripe phase to spatial variation

Consider perturbation in amplitude and phase (set $g=1$)

$$A(x,y,t) = \Big[\underbrace{\sqrt{\varepsilon-4q^2k^2}}_{A_0} + \underbrace{\rho(x,y,t)}_{\text{density fluctuation}}\Big]\, e^{ikx + i\underbrace{\phi(x,y,t)}_{\text{phase fluctuation}}}$$

linear order:

$$\partial_t A = \rho_t\, e^{ikx+i\phi} + A_0\, i\,\phi_t\, e^{ikx+i\phi}.$$

$$\partial_x A = \rho_x\, e^{ikx+i\phi} + (A_0+\rho)\, i(k+\phi_x)\, e^{ikx+i\phi}$$

$$\partial_y A = \rho_y\, e^{ikx+i\phi} + A_0\, i\,\phi_y\, e^{ikx+i\phi}$$

$$\partial_y^2 A = \rho_{yy}\, e^{ikx+i\phi} + \cancel{\rho_y \cdot i\phi_y\, e^{ikx+i\phi}}$$

$$+\, i\,\phi_{yy} A_0 e^{ikx+i\phi} - \cancel{A_0\, \phi_y^2\, e^{ikx+i\phi}}$$

$$\left(2q\frac{\partial}{\partial x} - i\frac{\partial^2}{\partial y^2}\right)A = (2q\rho_x - i\rho_{yy})\, e^{ikx+i\phi}$$

$$+ \Big[(A_0+\rho)2iq(k+\phi_x) + A_0\phi_{yy}\Big]\, e^{ikx+i\phi}$$

$$2q\frac{\partial}{\partial x}\cdot\left(2q\frac{\partial}{\partial x} - i\frac{\partial^2}{\partial y^2}\right)A$$

$$= \Big[(2q)^2\rho_{xx} - i2q\,\rho_{xyy} + 2iqk\,(2q\rho_x - i\rho_{yy})\Big]\, e^{ikx+i\phi}$$

$$+ \Big[4iq^2 k\rho_x + 4iq^2\phi_{xx}A_0 + 2q\phi_{xyy}A_0\Big]\, e^{ikx+i\phi}$$

$$+ \Big[(A_0+\rho)(2iq(k+\phi_x))^2 + 2iq(k+\phi_x)\cdot\phi_{yy}A_0\Big]\, e^{ikx+i\phi}$$

$$\frac{\partial}{\partial y}\left(2q\frac{\partial}{\partial x}-i\frac{\partial^2}{\partial y^2}\right)A = \left(2q\rho_{xy}-i\rho_{yyy}\right)e^{ikx+i\phi}$$
$$+\left[2iq\phi_{xy}+\phi_{yyy}\right]A_0 e^{ikx+i\phi}$$

$$-i\frac{\partial^2}{\partial y^2}\left(2q\frac{\partial}{\partial x}-i\frac{\partial^2}{\partial y^2}\right)A = \left(-2iq\rho_{xyy}-\rho_{yyyy}\right)e^{ikx+i\phi}$$
$$+\left[2q\phi_{xyy}-i\phi_{yyyy}\right]A_0 e^{ikx+i\phi}$$

$$\Rightarrow\left(2q\frac{\partial}{\partial x}-i\frac{\partial^2}{\partial y^2}\right)^2 A = \Big[(2q)^2\rho_{xx}-2iq\rho_{xyy}+2iqk(2q\rho_x-i\rho_{yy})$$
$$+A_0\left(4iq^2\phi_{xx}+2q\phi_{xyy}\right)+4iq^2k\rho_x$$
$$+(A_0+\rho)(-4q^2)(k^2+2k\phi_x)+2iqk\phi_{yy}A_0$$
$$+\left(-2iq\rho_{xyy}-\rho_{yyyy}\right)+\left(2q\phi_{xyy}-i\phi_{yyyy}\right)A_0\Big]e^{ikx+i\phi}$$

$$\frac{\partial A}{\partial t} = \varepsilon A + \left(2q\frac{\partial}{\partial x}-i\frac{\partial^2}{\partial y^2}\right)^2 A - |A|^2 A \text{ becomes}$$

$$\rho_t + A_0 i\phi_t = \varepsilon(A_0+\rho) - \overbrace{(A_0+\rho)^3}^{A_0^3+3A_0^2\rho+O(\rho^2)} + [\ldots]$$

- fluctuation in $\rho(x,y,t)$

$$\rho_t = \cancel{\varepsilon A_0 - A_0^3 - 4A_0 q^2 k^2}$$
$$+\varepsilon\rho - 3A_0^2\rho - 4q^2k^2\rho + (2q)^2\rho_{xx} + 2qk\rho_{yy}$$
$$+A_0 4q\phi_{xyy} - 4A_0 q^2(2k\phi_x) - \rho_{yyyy}$$
$$= (\varepsilon - 3A_0^2 - 4q^2k^2)\rho - 8A_0 q^2 k\phi_x$$
$$+ O(\rho_{xx}, \rho_{yy}, \phi_{xyy}, \rho_{yyyy})$$

$$\Rightarrow \quad \varepsilon - 3A_0^2 - 4q^2k^2 = A_0^2 - 3A_0^2 = -2A_0^2$$

$$\rho_t \simeq -2A_0^2\left(\rho + \frac{4q^2k}{A_0}\phi_x\right) + O(\rho_{xx}, \rho_{yy}, \cdots)$$

$$\rightarrow \quad \rho \simeq -\frac{4q^2k}{A_0}\phi_x \qquad (\rho_t \ll \rho \text{ ; see below})$$

- fluctuation in $\phi(x,y,t)$

$$A_0\phi_t = 2(2q)^2 k\rho_x + A_0(4q^2)\phi_{xx} + 2qk\,\phi_{yy}A_0$$
$$-4q\rho_{xyy} - \phi_{yyyy}A_0$$

use $\rho_x = -\frac{4q^2k}{A_0}\phi_{xx}$

$$\rightarrow \quad \phi_t = (4q^2)\cdot\left[1 - \frac{2k^2\cdot 4q^2}{A_0^2}\right]\phi_{xx} + 2qk\phi_{yy}$$
$$+ O(\rho_{xyy}, \phi_{yyyy})$$

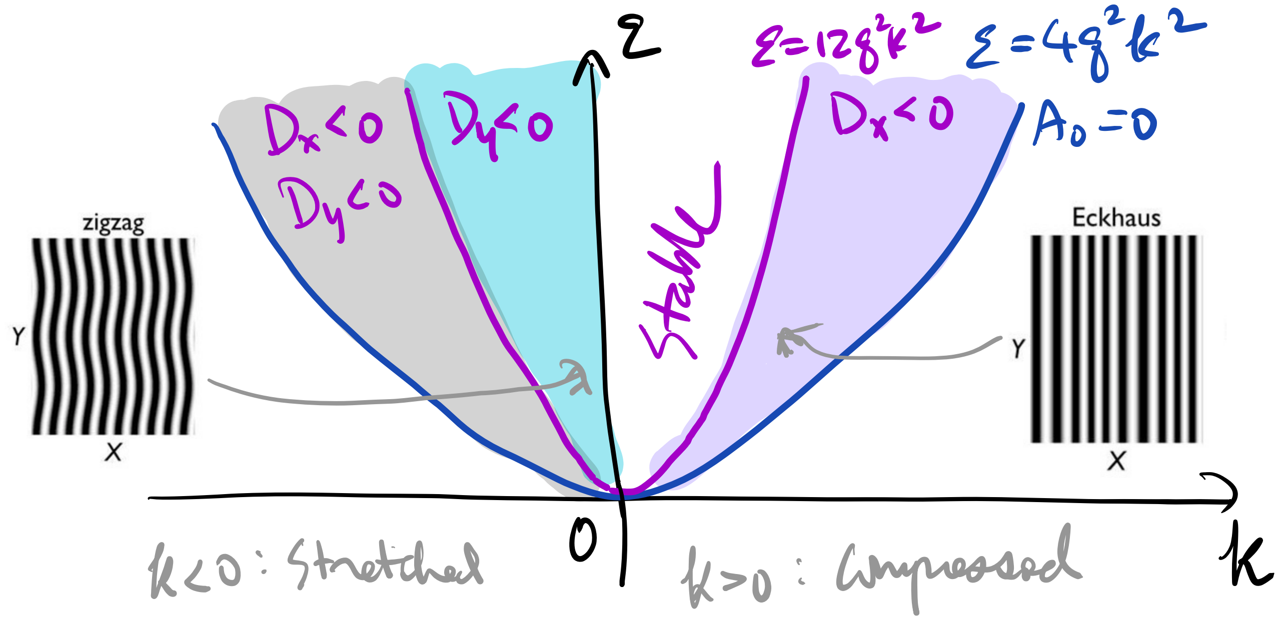
Anisotropic diffusion eqn for phase field:

$$\phi_t = D_x\phi_{xx} + D_y\phi_{yy} + O(\phi_{xxyy}, \phi_{yyyy})$$

$$D_x = 4q^2\cdot\frac{\varepsilon - 12q^2k^2}{\varepsilon - 4q^2k^2} \text{ ; } D_y = 2qk$$

Instability if $D_x < 0$ (Eckhaus instability)

or $D_y < 0$. (Zigzag instability)

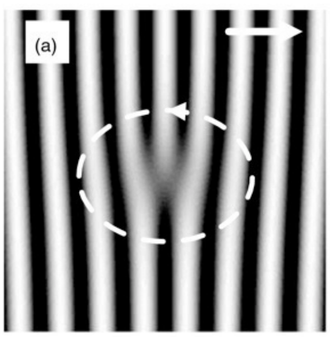


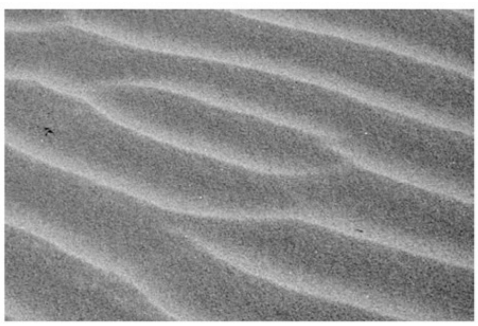
$\Rightarrow$ slightly compressed stripes are stable

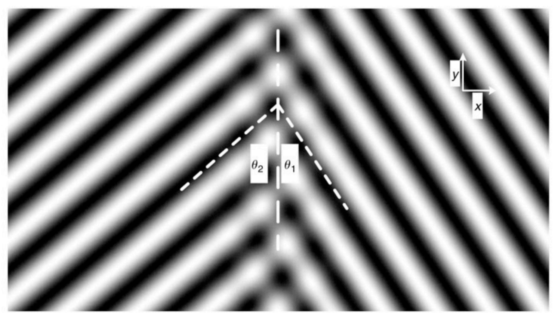
$\Rightarrow$ Still subject to additional "defects"

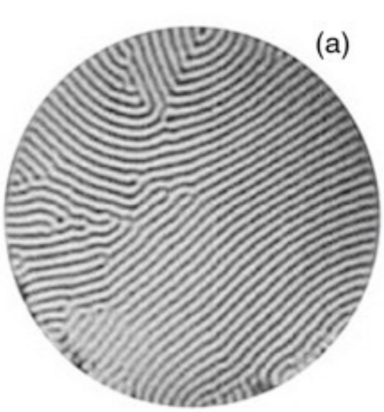
dislocations and their interaction

grain boundary









Stability of "structure" from underlying energy function:

$$\frac{\partial u}{\partial t} = -\frac{\delta}{\delta u}\mathcal{H};$$

$$\mathcal{H} = \int d^2r \left[-\frac{\varepsilon}{2}u^2 + \frac{g}{12}u^4 + \frac{1}{2}\left[(q^2+\nabla^2)u\right]^2 \right]$$

c) Pattern Selection in 2d:

Stability of stripes against array of spots?

Spots as <u>superposition</u> of stripes

let $$u(\vec{r},t) = \sum_{l=1}^{m} \left(A_l(t)\, e^{i\vec{q}_l\cdot\vec{r}} + A_l^*(t)\, e^{-i\vec{q}_l\cdot\vec{r}} \right)$$

where $\vec{q}_l = q\,\hat{n}_l$ and $\hat{n}_l^2 = 1$

↑ unit vector in direction of l^{th} stripe

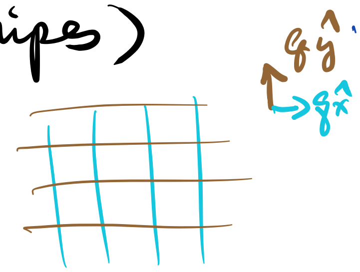
in 2d: $m=1 \to$ stripes

$m=2 \to$ square or rhomboid

$m=3 \to$ hexagonal

✗ Square vs Stripes:

$m=2$: Square pattern (orthogonal stripes)



let $$u(x,y,t) = A_1(t)e^{iqx} + A_1^*(t)e^{-iqx} + A_2(t)e^{iqy} + A_2^*(t)e^{-iqy}$$

(Same as $u = a_1\cos qx + b_1 \sin qx + a_2 \cos qy + b_2 \sin qy$)

$$\begin{aligned} u^2 = {} & A_1^2 e^{2iqx} + A_1^{*2} e^{-2iqx} + 2|A_1|^2 \\ & + A_2^2 e^{2iqy} + A_2^{*2} e^{-2iqy} + 2|A_2|^2 \\ & + 2A_1A_2 e^{iqx+iqy} + 2A_1^*A_2^* e^{-iqx-iqy} \\ & + 2A_1A_2^* e^{iqx-iqy} + 2A_1^*A_2 e^{-iqx+iqy} \end{aligned}$$

$$u^3 = 2(|A_1|^2+|A_2|^2)\cdot u + |A_1|^2 A_1 e^{iqx} + |A_1|^2 A_1^* e^{-iqx}$$
$$+ |A_2|^2 A_2 e^{iqy} + |A_2|^2 A_2^* e^{-iqy}$$
$$+ 2A_1|A_2|^2 e^{iqx} + 2|A_1|^2 A_2 e^{iqy}$$
$$+ 2|A_1|^2 A_2^* e^{-iqy} + 2A_1^*|A_2|^2 e^{-iqx}$$
$$+ 2|A_1|^2 A_2^* e^{-iqy} + 2A_1|A_2|^2 e^{iqx}$$
$$+ 2|A_1|^2 A_2 e^{iqy} + 2A_1^*|A_2|^2 e^{-iqx}$$
$$+ \mathcal{O}(A_1^3 e^{3iqx}, A_1^2 A_2 e^{2iqx+iqy}, \text{etc})$$

$$u^3 = 3(|A_1|^2+|A_2|^2)(A_1 e^{iqx}+A_1^* e^{-iqx}+A_2 e^{iqy}+A_2^* e^{-iqy})$$
$$+ 3|A_1|^2(A_2 e^{iqy}+A_2^* e^{-iqy}) + 3|A_2|^2(A_1 e^{iqx}+A_1^* e^{-iqx})$$
+ higher harmonics.

Insert into Swift-Hohenberg eqn

$$\partial_t u = [\varepsilon - (q^2+\nabla^2)^2]u - \tfrac{1}{3}u^3$$

$$\rightarrow \begin{cases} \dot{A}_1 = \varepsilon A_1 - |A_1|^2 A_1 - 2|A_2|^2 A_1 \\ \dot{A}_2 = \varepsilon A_2 - |A_2|^2 A_2 - 2|A_1|^2 A_2 \end{cases}$$

2-variable dynamical system!

fixed points: $(\bar{A}_1, \bar{A}_2)$

$$\varepsilon \bar{A}_1 = \bar{A}_1(|\bar{A}_1|^2 + 2|\bar{A}_2|^2)$$
$$\varepsilon \bar{A}_2 = \bar{A}_2(|\bar{A}_2|^2 + 2|\bar{A}_1|^2)$$

Sol'n: 1) $\bar{A}_1 = 0, \bar{A}_2 = 0$. (no pattern)

2) $\bar{A}_1 = 0 \rightarrow \varepsilon \bar{A}_2 = \bar{A}_2 |A_2|^2 . \rightarrow |A_2|^2 = \varepsilon .$

$A_2 = \sqrt{\varepsilon}\, \underbrace{e^{i\varphi_2}}_{\text{shift of phase along } y}$ stripes in y-direction

3) $\bar{A}_1 = \sqrt{\varepsilon}\, e^{i\varphi_1}, \bar{A}_2 = 0$ stripes in x-direction

4) $\bar{A}_1 \neq 0, \bar{A}_2 \neq 0$ square array

$$\begin{aligned} \varepsilon &= |A_1|^2 + 2|A_2|^2 \\ \varepsilon &= |A_2|^2 + 2|A_1|^2 \end{aligned} \rightarrow \begin{aligned} A_1 &= \sqrt{\varepsilon/3}\, e^{i\varphi_1} \\ A_2 &= \sqrt{\varepsilon/3}\, e^{i\varphi_2} \end{aligned}$$

- Stability of these fixed points?

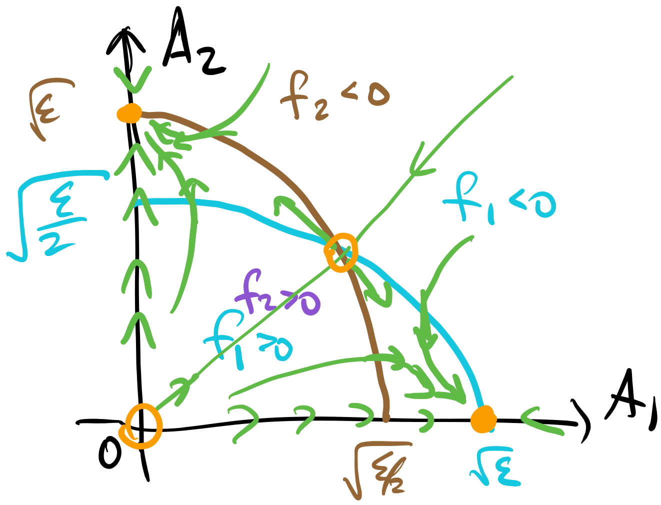
treat A_i as real (= fixing phase)

let $\dot{A}_1 = A_1 f_1(A_1, A_2)$

$\dot{A}_2 = A_2 f_2(A_1, A_2)$

$f_1 = \varepsilon - A_1^2 - 2A_2^2$

$f_2 = \varepsilon - A_2^2 - 2A_1^2$

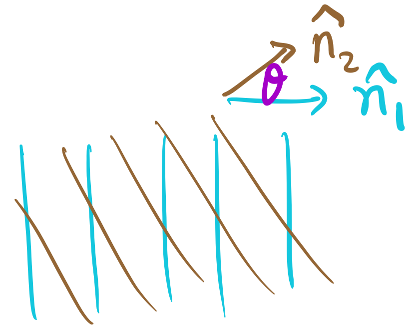


$\Rightarrow$ Square array unstable to stripes

What about a general rhomboid?

$$u(\vec{r}, t) = \sum_{\ell=1}^{2} \left(A_\ell(t)\, e^{i \vec{q}_\ell \cdot \vec{r}} + A_\ell^*(t)\, e^{-i \vec{q}_\ell \cdot \vec{r}} \right)$$

$$\vec{q}_\ell = q\, \hat{n}_\ell$$



General form of the amplitude eqn:

$$\frac{\partial A_1}{\partial t} = \varepsilon A_1 - \left(|A_1|^2 + \gamma(\theta)|A_2|^2\right) A_1$$

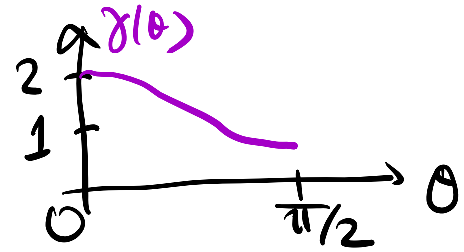
$$\frac{\partial A_2}{\partial t} = \varepsilon A_2 - \left(|A_2|^2 + \gamma(\theta)|A_1|^2\right) A_2$$

- form of $\gamma(\theta)$ depends on details of nonlinear interaction

e.g. for the generalized Swift-Hohenberg model

$$\frac{\partial u}{\partial t} = \left[\varepsilon - (q^2+\nabla^2)^2\right] u - \frac{g_1}{3} u^3 + g_3 (\nabla u)^2 \nabla^2 u$$

$$\gamma(\theta) = 2 - \frac{2}{3}\frac{g_3}{g_1+g_3}(1-\cos 2\theta)$$



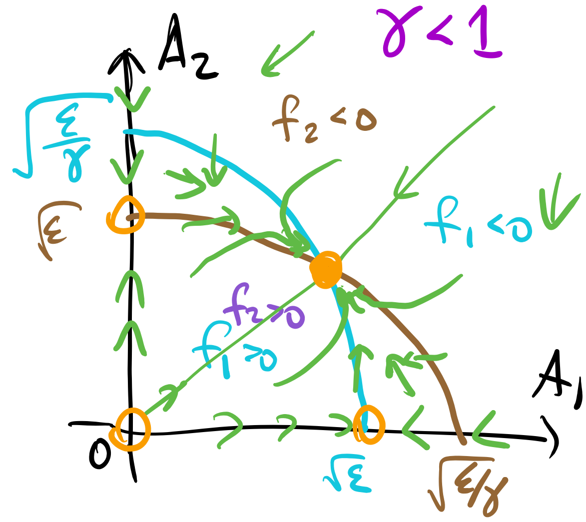
Analyze stability for general γ:

$$\dot{A}_1 = A_1 f_1(A_1, A_2)$$

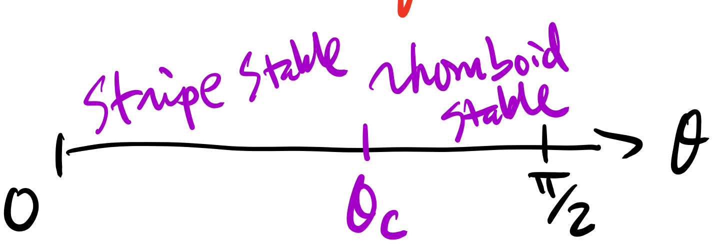
$$\dot{A}_2 = A_2 f_2(A_1, A_2)$$

$$f_1 = \varepsilon - A_1^2 - \gamma A_2^2$$

$$f_2 = \varepsilon - A_2^2 - \gamma A_1^2$$



$\Rightarrow$ rhomboid/square phase stabilized if $\gamma(\theta) < 1$.



For generalized SH model $\theta_c < \pi/2$ if $g_3 > 3 g_1$

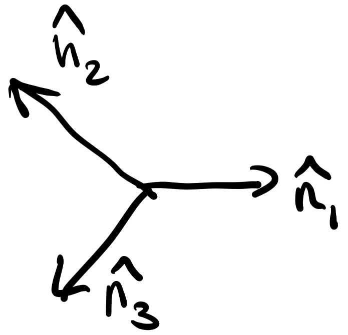
* hexagon vs stripe

hexagon ($n=3$): $\vec{q}_\ell = q \cdot \hat{n}_\ell$

$\hat{n}_1 = \hat{x}$,

$\hat{n}_2 = -\frac{1}{2}\hat{x} + \frac{\sqrt{3}}{2}\hat{y}$

$\hat{n}_3 = -\frac{1}{2}\hat{x} - \frac{\sqrt{3}}{2}\hat{y}$



$$
\begin{aligned}
u(\vec{r},t) = \; & A_1(t)e^{i\vec{q}_1\cdot\vec{r}} + A_1^*(t)e^{-i\vec{q}_1\cdot\vec{r}} \\
& + A_2(t)e^{i\vec{q}_2\cdot\vec{r}} + A_2^*(t)e^{-i\vec{q}_2\cdot\vec{r}} \\
& + A_3(t)e^{i\vec{q}_3\cdot\vec{r}} + A_3^*(t)e^{-i\vec{q}_3\cdot\vec{r}}
\end{aligned}
$$

Consider the more general Haken model (w/o $u \to -u$ symmetry)

$$\frac{\partial u}{\partial t} = \left[\varepsilon - (q^2 + \nabla^2)^2\right] u + v u^2 - g u^3$$

Set $v=1$, $g=1$ (amts to rescaling u + t)

($v>0$: H0 hexagon; $v<0$: Hπ)

amplitude eqn:

$\gamma(\theta = 2\pi/3)$

$$\dot{A}_1 = \varepsilon A_1 + A_2^* A_3^* - \left[|A_1|^2 + \gamma\left(|A_2|^2 + |A_3|^2\right)\right] A_1$$

$$\dot{A}_2 = \varepsilon A_2 + A_3^* A_1^* - \left[|A_2|^2 + \gamma\left(|A_3|^2 + |A_1|^2\right)\right] A_2$$

$$\dot{A}_3 = \varepsilon A_3 + A_1^* A_2^* - \left[|A_3|^2 + \gamma\left(|A_1|^2 + |A_2|^2\right)\right] A_3$$

Use $A_\ell = R_\ell e^{i\phi_\ell}$

get $\frac{d}{dt}(\phi_1+\phi_2+\phi_3) = -\# \sin(\phi_1+\phi_2+\phi_3)$

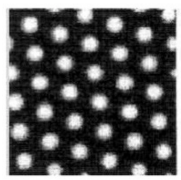
$\to \phi_1+\phi_2+\phi_3 = 0$ (Stable)

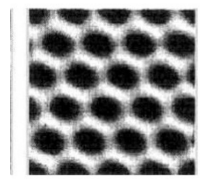
$$\to \quad \dot{R}_1 = \varepsilon R_1 + R_2 R_3 - R_1^3 - \gamma R_1 (R_2^2 + R_3^2)$$
$$\dot{R}_2 = \varepsilon R_2 + R_3 R_1 - R_2^3 - \gamma R_2 (R_3^2 + R_1^2)$$
$$\dot{R}_3 = \varepsilon R_3 + R_1 R_2 - R_3^3 - \gamma R_3 (R_1^2 + R_2^2)$$

nontrivial fixed pts:

i) $R_1 = R_2 = R_3 \equiv R^H$ (hexagonal)

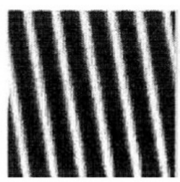
$$R^H = \frac{1 \pm \sqrt{1+4\varepsilon(1+2\gamma)}}{2(1+2\gamma)}$$

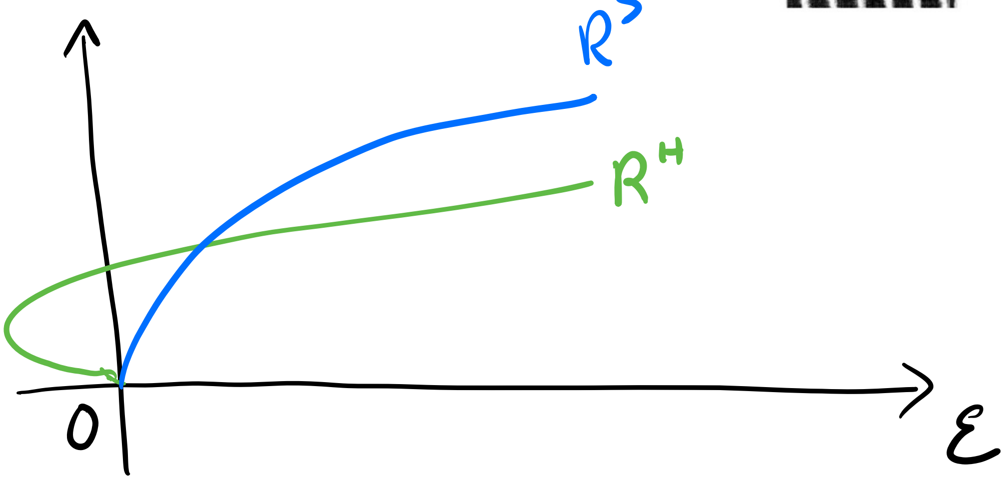

H0 $(\nu > 0)$


Hπ $(\nu < 0)$

ii) $R_\ell = R^S$, $R_{\ell' \neq \ell} = 0$ (Stripe)

$$R^S = \sqrt{\varepsilon}$$





Next analyse stability:

$$R_\ell(t) = \bar{R}_\ell + \delta R_\ell(t) \rightarrow \delta\dot{R}_\ell = M_{\ell\ell'}\,\delta R_{\ell'}$$

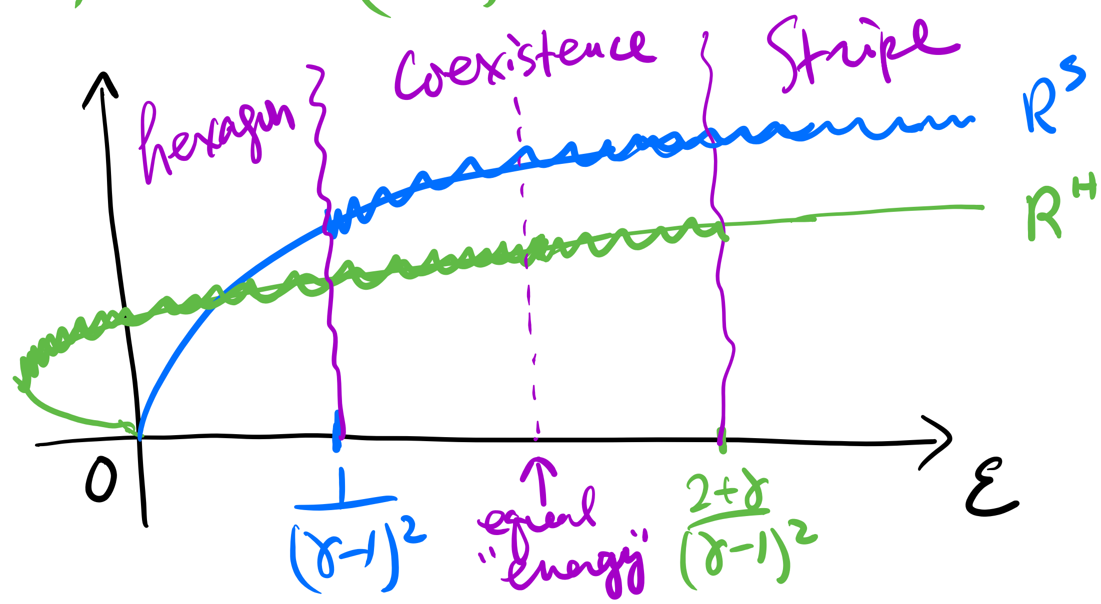
$$M^R = \begin{pmatrix} a & b & b \\ b & a & b \\ b & b & a \end{pmatrix} \qquad M^S = \begin{pmatrix} -2\varepsilon & 0 & 0 \\ 0 & (1-\gamma)\varepsilon & \sqrt{\varepsilon} \\ 0 & \sqrt{\varepsilon} & (1-\gamma)\varepsilon \end{pmatrix}$$

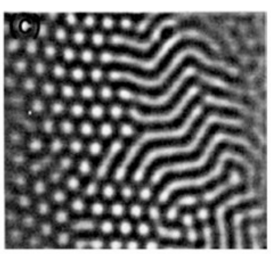
$$a = \varepsilon - (3+2\gamma)(R^H)^2$$

$$b = R^H - 2\gamma (R^H)^2$$

Stable range:

$$-\frac{1}{4(1+2\gamma)} < \varepsilon < \frac{(2+\gamma)}{(\gamma-1)^2} \qquad \varepsilon > \frac{1}{(1-\gamma)^2}$$





hexagon-stripe coexistence

Slow invasion by the more stable state