Tutorial TA session

PHYS 239 – Spatiotemporal Dynamics in Biological Systems

Leonardo Pacciani-Mori
January 10th, 2022

UC San Diego

Topics covered by this tutorial

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- Taylor expansion

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- First-order linear ODEs and coupled linear ODEs

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- First-order linear ODEs and coupled linear ODEs
- Linear stability analysis

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- Bifurcation theory

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Important!

Please ***DO*** interrupt me at *any time* if you have questions!

Taylor expansion

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Basic idea

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We want to approximate a function $f(x)$ around a point x_0 using polynomials.
Why? Because polynomials are simple! This way, we can express complicated functions with simpler terms (at least locally).

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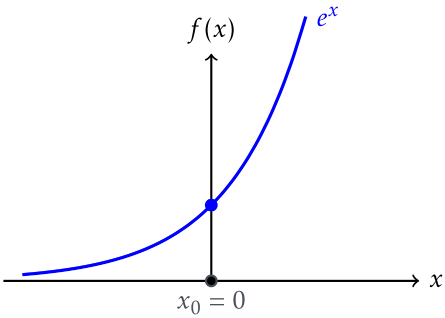

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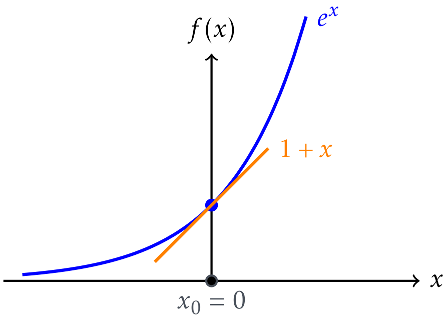

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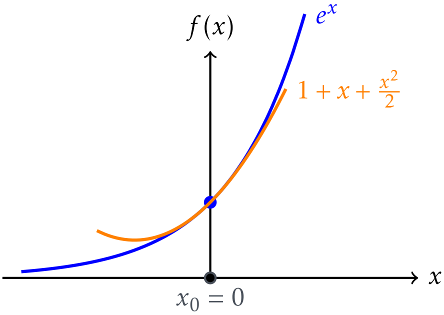

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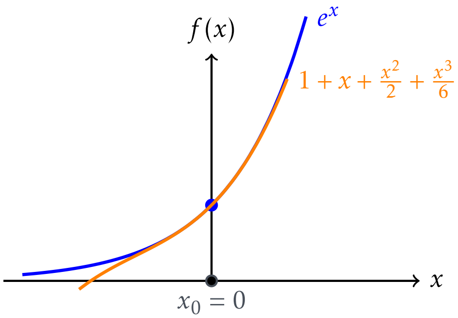

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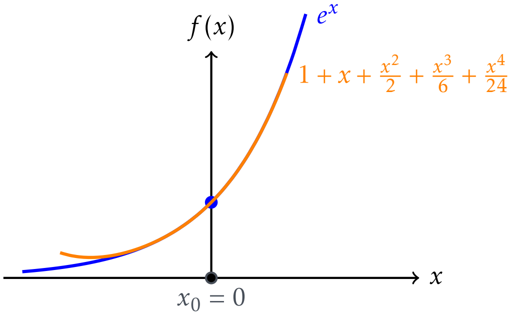

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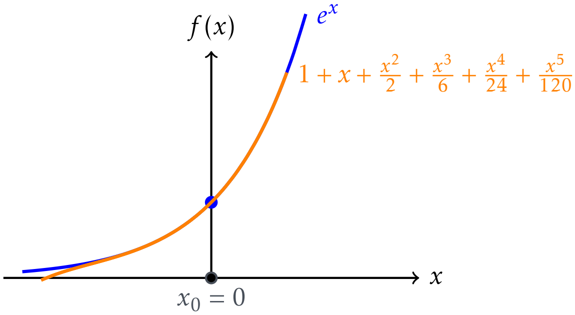

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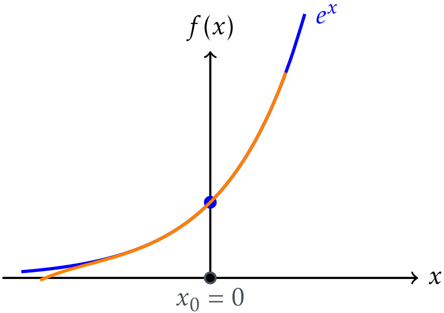

Taylor's theorem

If $f(x)$ is continuous (i.e., it doesn't have any "jumps") and differentiable (i.e., it doesn't have "cusps" or "spikes") in x_0, then around x_0 we can approximate:

$$
\begin{aligned}
f(x) \approx f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2} f''(x_0)(x - x_0)^2 + \\
+ \frac{1}{6} f'''(x_0)(x - x_0)^3 + \cdots \quad (1)
\end{aligned}
$$

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- In practice, you will never need to go beyond the second order term.
  If you do, there is almost certainly something wrong: all the interesting physics happens within the first two orders, beyond that it's just computational subtleties that we will never be interested in

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  If we are close enough to x_0, we can approximate *any* function with a linear one

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- The most important point from this theorem is the following:
  If we are close enough to x_0, we can approximate *any* function with a linear one

This last point is particularly important, because we can say a lot about linear ODE but almost *nothing* about **non**-linear ODEs in general. Thanks to this theorem, however, we can understand the behavior of nonlinear ODEs around specific points.

First-order linear ODEs

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$$\frac{dy(t)}{dt} = \lambda y(t) \tag{2}$$

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This is just a trick and not the mathematically rigorous way to show that the solution of eq 2 is an exponential. But it works, so we use it.

First-order linear coupled ODEs

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A system of couple ODEs are two or more ODEs where each variable depends on the other ones. For example:

$$\frac{dx}{dt} = a_1 x + b_1 y \qquad \frac{dy}{dt} = a_2 x + b_2 y \tag{3}$$

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The trick, in this case, is to use the matrix notation to rewrite the equation in terms of one two-dimensional variables:

$$\Rightarrow \quad \frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \underbrace{\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}}_{:=A}\begin{pmatrix} x \\ y \end{pmatrix} = A\begin{pmatrix} x \\ y \end{pmatrix} \tag{4}$$

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We can now use the same trick shown before to write the solution of eq (3):

$$\begin{pmatrix} \boldsymbol{x(t)} \\ \boldsymbol{y(t)} \end{pmatrix} = \mathbf{exp}(\boldsymbol{At})\begin{pmatrix} \boldsymbol{x(0)} \\ \boldsymbol{y(0)} \end{pmatrix} \tag{5}$$

even though it is not clear what is the meaning of the exponential of a matrix. We are also assuming $t_0 = 0$ for the sake of simplicity.

First-order linear coupled ODEs

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For our purposes we actually don't need to know what the exponential of a matrix is. A mathematical theorem, in fact, ensures us that we can write the solution of our coupled ODEs as:

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 e^{\lambda_1 t} \vec{u}_1 + c_2 e^{\lambda_2 t} \vec{u}_2 \tag{7}$$

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- $\vec{u}_1$ and $\vec{u}_2$ are the two *eigenvectors* of the matrix A
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Therefore

In order to solve a system of coupled linear ODEs we just need to know the so-called *spectral properties* (i.e., eigenvalues and eigenvectors) of the matrix A

First-order linear coupled ODEs

As an example, let's solve this system:

$$\frac{dx}{dt} = 3x - 4y \qquad \frac{dy}{dt} = 4x - 7y \tag{8}$$

with initial conditions $x(0) = y(0) = 1$.

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where $\mathbb{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is the identity matrix.

First-order linear coupled ODEs

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Therefore, we need to solve:

$$\det\left[\begin{pmatrix} 3 & -4 \\ 4 & -7 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right] = \det\begin{pmatrix} 3-\lambda & -4 \\ 4 & -7-\lambda \end{pmatrix} = 0 \tag{12}$$

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Since by definition:

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc \tag{13}$$

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First-order linear coupled ODEs

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First-order linear coupled ODEs

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First-order linear coupled ODEs

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We can now use the eigenvalues to find the eigenvectors.

First-order linear coupled ODEs

By definition the eigenvectors $\vec{u}_1$ and $\vec{u}_2$ satisfy $A\vec{u}_1 = \lambda_1\vec{u}_1$ and $A\vec{u}_2 = \lambda_2\vec{u}_2$, respectively.

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First-order linear coupled ODEs

$$\frac{dx}{dt} = 3x - 4y \qquad \frac{dy}{dt} = 4x - 7y \quad \text{with } x(0) = y(0) = 1 \qquad \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 e^{\lambda_1 t} \vec{u}_1 + c_2 e^{\lambda_2 t} \vec{u}_2 \tag{21}$$

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Therefore, the solution of our system of coupled ODEs is:

$$x(t) = \frac{2}{3} e^{t} + \frac{1}{3} e^{-5t} \qquad y(t) = \frac{1}{3} e^{t} + \frac{2}{3} e^{-5t} \tag{24}$$

Linear stability analysis

We would like to solve non-linear ODEs (and systems of coupled ODEs), i.e.:

$$\dot{x} = f(x) \tag{25}$$

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$$\dot{x} = x - x^2 \qquad \dot{x} = \frac{x}{1+x} \tag{26}$$

Problem

Contrarily to linear ODEs, there is no general theorem that allows us to solve non-linear ODEs in general. They almost always *cannot* be solved analytically.

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Can we say anything about these systems without solving them analytically?

We can use Taylor expansion to approximate the non-linear function $f(x)$ with a linear one around a point of interest x_0, and solve the linearized ODEs.

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Linear stability analysis

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The most interesting points of any dynamical system are its *equilibria* (or *stationary points*):

$$\dot{x} = f(x) \quad \Rightarrow \quad x^* \text{ is an equilibrium if } f(x^*) = 0 \tag{27}$$

"Informal" definitions

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We can get a sense of the stability of equilibria without even trying to solve a non-linear system by drawing *stream plots* (or *flow plots*).

Stream plots

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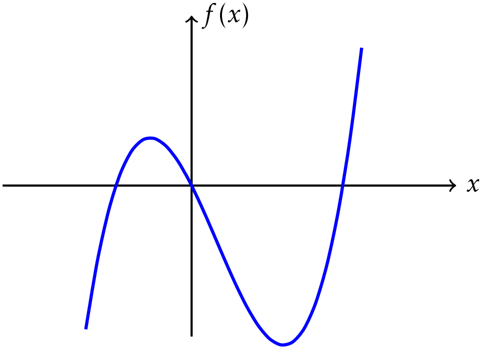

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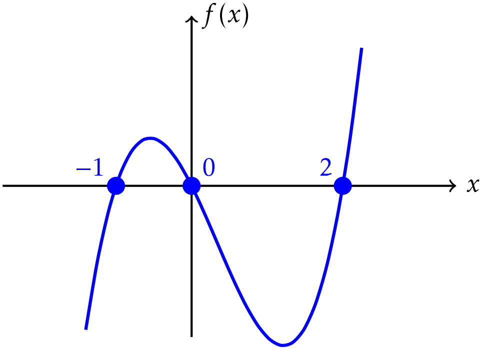

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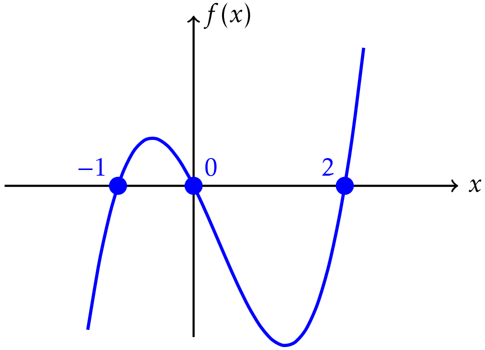

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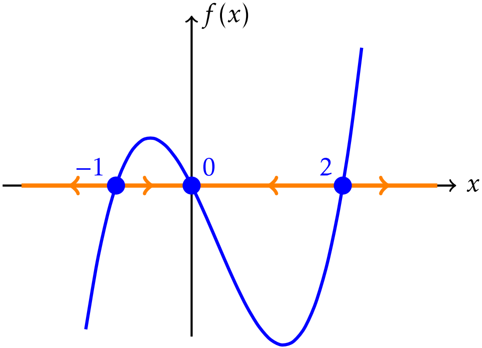

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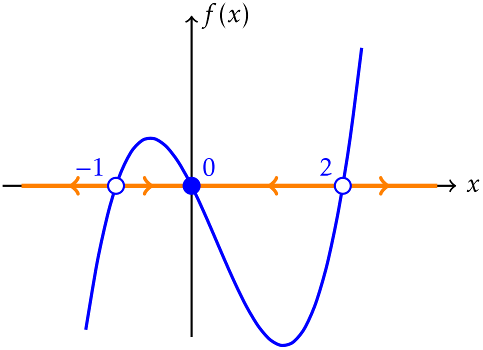

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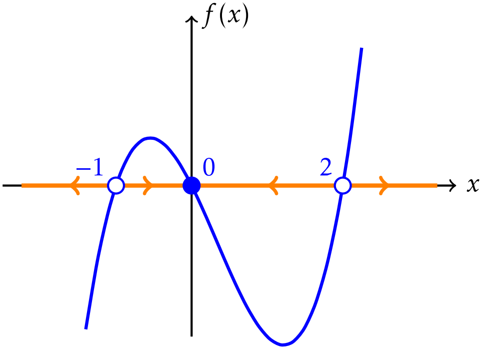

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Important

In one dimension, if there is more than one equilibrium their stability always "alternates": after a stable equilibrium we must find an unstable one, and viceversa.

Linear stability analysis

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If $f(\vec{x}_0) = 0$ (i.e., $\vec{x}_0$ is an equilibrium) we can use Taylor's theorem to approximate the system around $\vec{x}_0$ as:

$$\dot{x} = \underbrace{f(\vec{x}_0)}_{=0} + J(\vec{x}_0)(\vec{x} - \vec{x}_0) = J(\vec{x}_0)(\vec{x} - \vec{x}_0) \tag{30}$$

Linear stability analysis

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$$\vec{x}(t) = \vec{x}_0 + \sum_{i=1}^{n} c_i e^{\lambda_i t} \vec{u}_i \tag{33}$$

where λ_i are the eigenvalues of $J(\vec{x}_0)$, $\vec{u}_i$ are its eigenvectors, and c_i are constants (to be determined from the initial conditions).

Linear stability analysis

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- If some $\mathrm{Re}\,\lambda_i < 0$ and some $\mathrm{Re}\,\lambda_i = 0$, this method does not allow to determine the stability of the equilibrium

Linear stability analysis

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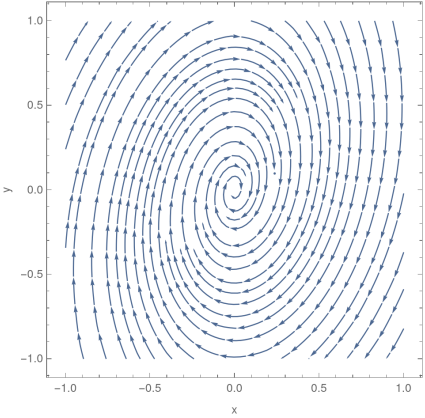

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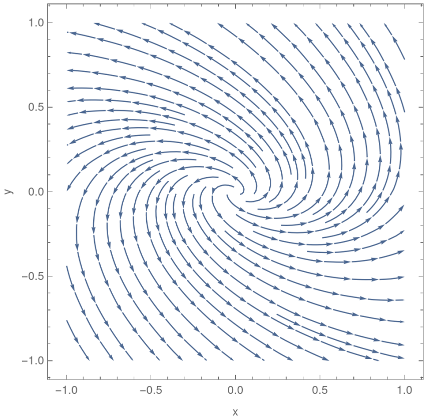

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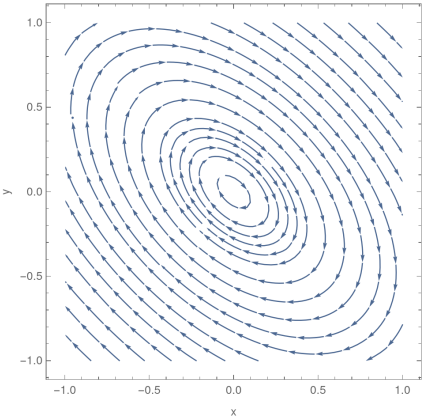

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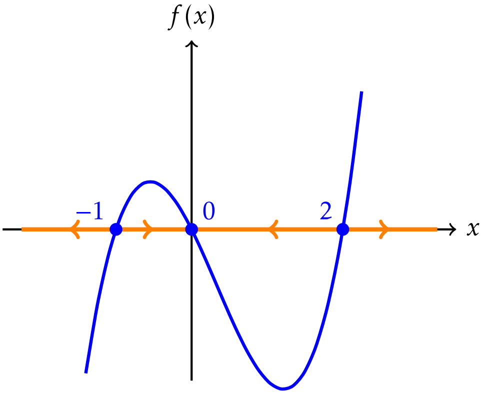

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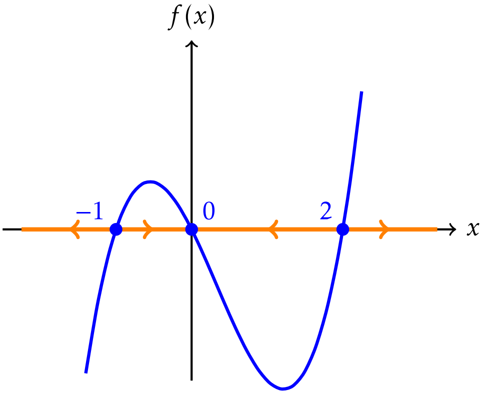

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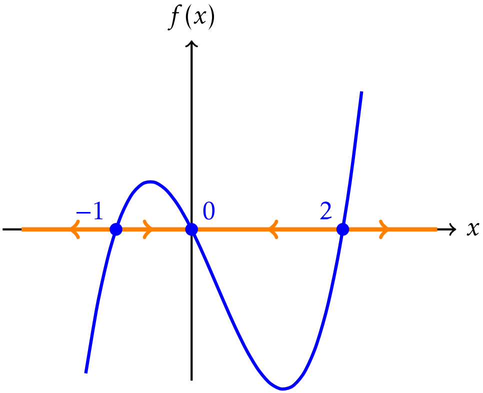

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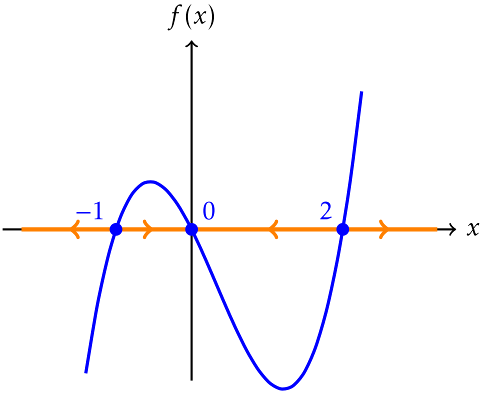

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Linear stability analysis

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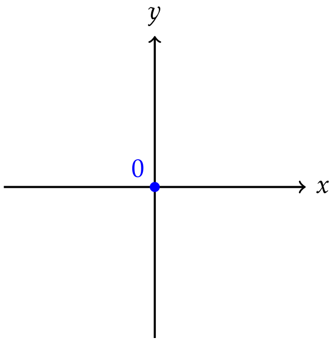

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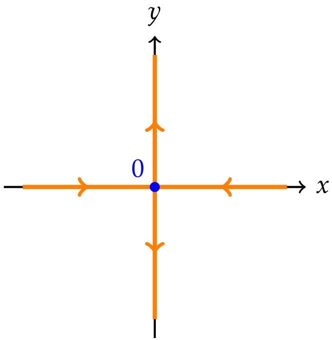

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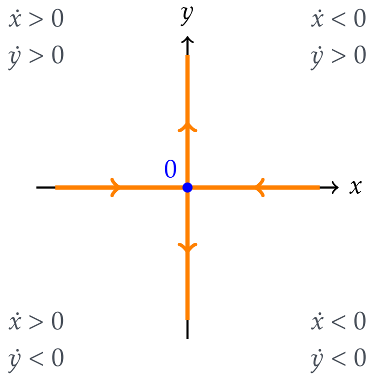

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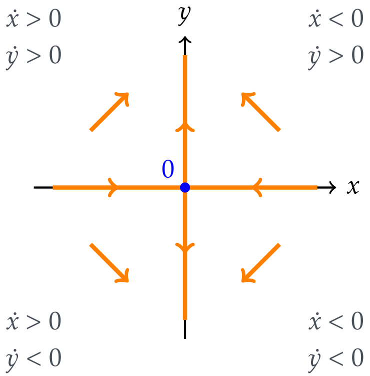

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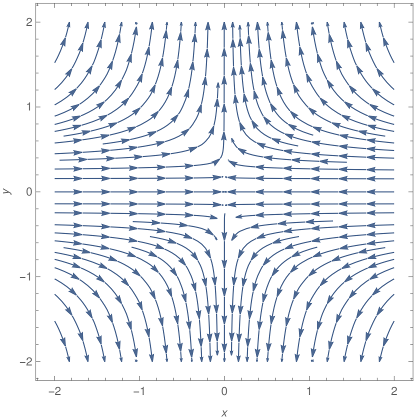

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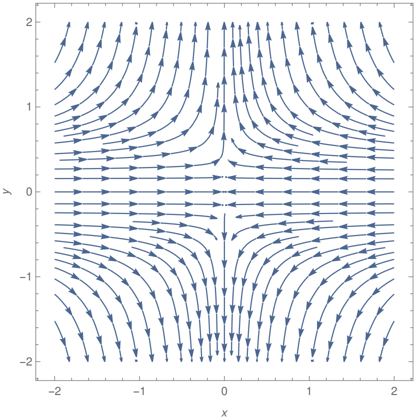
What if we draw the stream plot?

The equilibrium is ***unstable***.
In this case $(0, 0)$ is called ***saddle point***.



Linear stability analysis

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$$\dot{x} = -x \qquad \dot{y} = ky^3 \qquad \text{with} \quad k > 0 \tag{37}$$

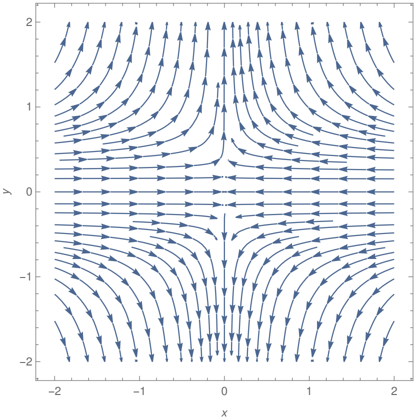
$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x \\ ky^3 \end{pmatrix} \tag{38a}$$

The only equilibrium is $(x, y) = (0, 0)$, and the jacobian matrix is:

$$J = \begin{pmatrix} -1 & 0 \\ 0 & 3ky^2 \end{pmatrix}_{|y=0} = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \tag{38b}$$

Therefore, the eigenvalues are -1 and 0 $\longrightarrow$ we can't say anything about the equilibrium!

What if we draw the stream plot?



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Excercise: what happens if $k < 0$?

Bifurcation theory

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Bifurcation theory

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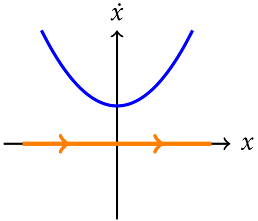

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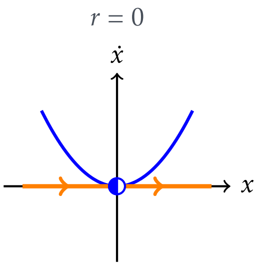

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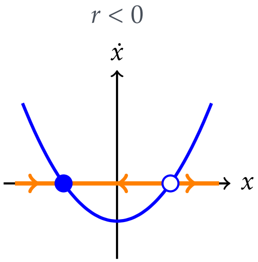

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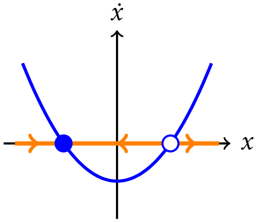
The bifurcation point in this system is $r = 0$.

$r < 0$



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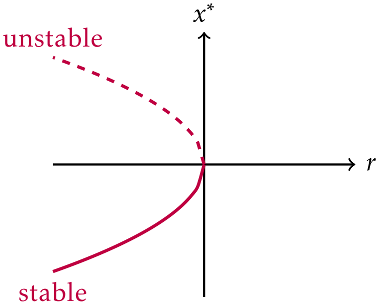

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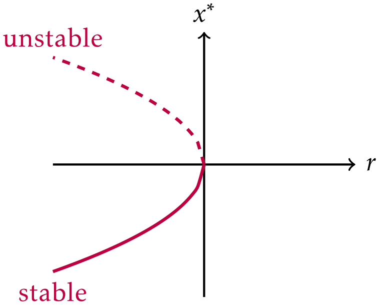

Important

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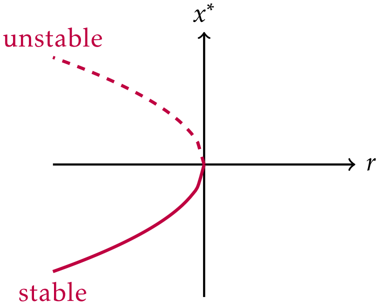

Important

This type of bifurcation is also known as *turning point bifurcation*, because the bifurcation point $r = 0$ can also be called *turning point*

Bifurcation theory

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The function is a parabola with roots at $x^* = 0$ and $x^* = r$, maximum in $x_{\mathrm{max}} = r/2$ and its maximum value is $f(x_{\mathrm{max}}) = r^2/4$.

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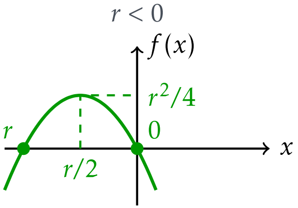

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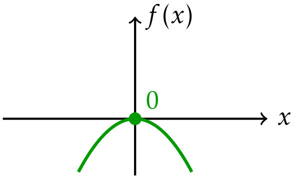

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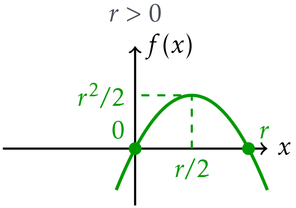

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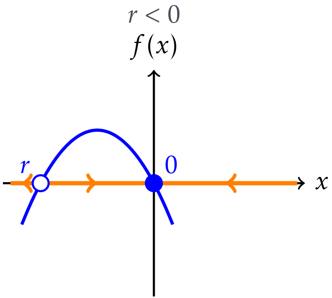

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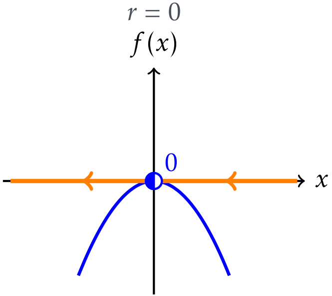

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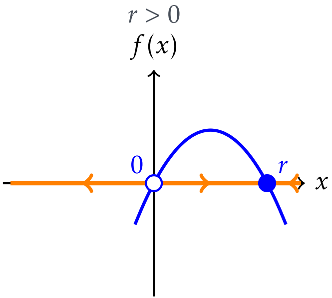

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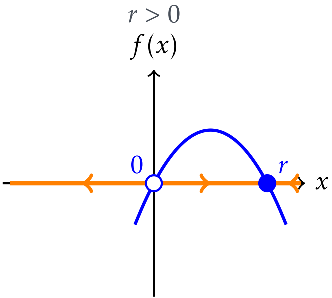

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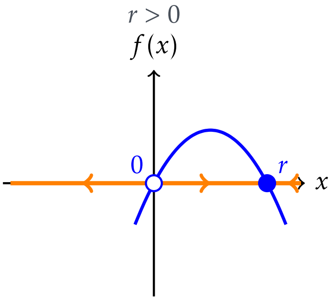

Therefore

One equilibrium always remains in $x^* = 0$, but as r is changed another equilibrium ($x^* = r$) "crosses" over it and "exchanges stability" with it.

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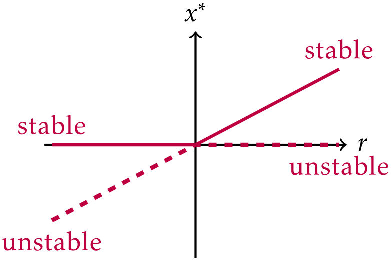

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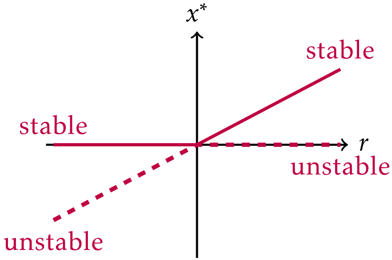

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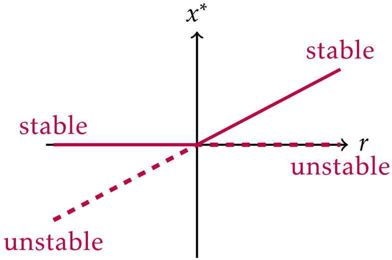

Important

Contrarily to the saddle-point bifurcation no equilibrium is created or destroyed in this case, but the bifurcation leads to the stability being “exchanged” between equilibria.

Bifurcation theory

Supercritical pitchfork bifurcation

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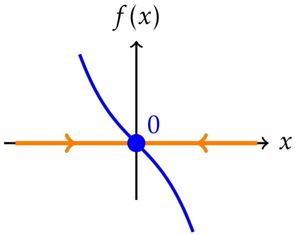

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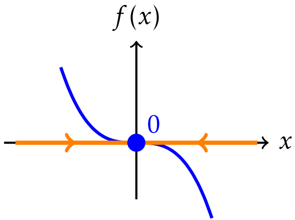

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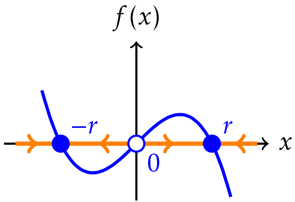

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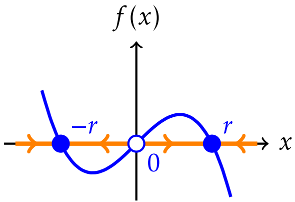

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A stable equilibrium becomes unstable and two new stable equilibria branch out of it.

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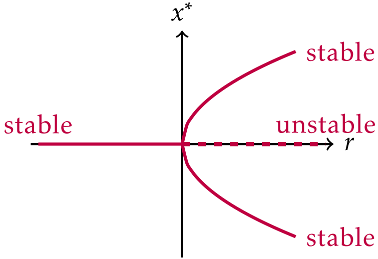

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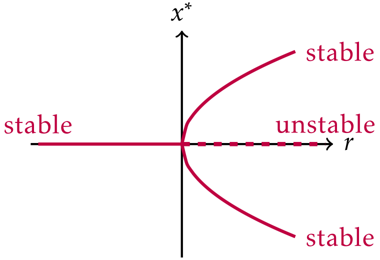

(this is also why these types of bifurcations are called *pitchfork bifurcations*).

Bifurcation theory

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Excercise

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Excercise

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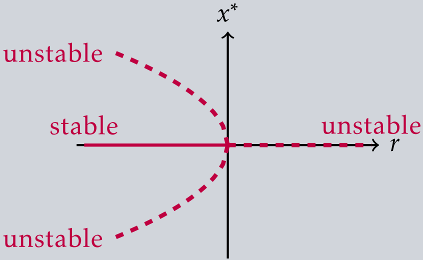
Draw a streamplot of the system for $r < 0$, $r = 0$ and $r > 0$ and verify that the bifurcation diagram looks like this:



Bifurcation theory

Last remarks

Since the properties of a bifurcation depend on ODEs $\dot{x} = f(x)$ here $f(x)$ is a simple polynomial, the results we've found are always true for *any* $f(x)$ locally around a point x_0, using Taylor's expansion.

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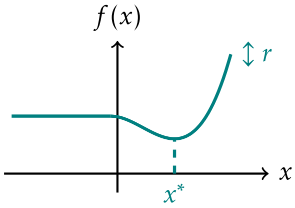

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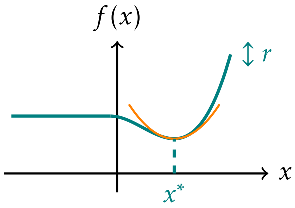

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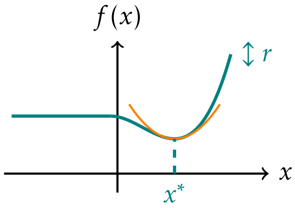

Locally around x^*, $f(x)$ "looks like" a parabola of the form $r + x^2$, so we know that as r changes the system will exhibit a *saddle-point bifurcation* around x^*.

That's all!

Questions?

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