B3. phase diagram and feasibility space

a) Ecological phase diagram (for 2 species)

Strong orthogonal nutrient pref
broad coexist. regime

Similar nut. pref.
narrow coexist. regime

⇒ phenotypical landscape
for fixed environment \((n_0^n, n_0^B)\)

**Algebraic analysis:**

\[
\begin{bmatrix}
P_1^*

P_2^*

\end{bmatrix} =
\begin{bmatrix}
\gamma_{2B} \frac{n_0^n - n_0^B}{n_0^n - n_0^B} \gamma_A - \gamma_{2A} \frac{n_0^B n_0^A}{n_0^n - n_0^B} \gamma_B

-\gamma_{1B} \frac{n_0^B - n_0^n}{n_0^B - n_0^n} \gamma_A + \gamma_{1A} \frac{n_0^n - n_0^B}{n_0^n - n_0^B} \gamma_B

\end{bmatrix} =
\begin{bmatrix}
\frac{\gamma_{2B} n_0^n}{\gamma_{2B} - \gamma_{1B} \gamma_{1A}} \frac{\gamma_A}{\gamma_A - \gamma_{2A} \gamma_A} \gamma_A - \gamma_{2A} \gamma_B

-\gamma_{1B} \frac{\gamma_A}{\gamma_A - \gamma_{2A} \gamma_A} \gamma_A + \gamma_{1A} \frac{\gamma_A}{\gamma_A - \gamma_{2A} \gamma_A} \gamma_B

\end{bmatrix}
\]

Where \(j^x = \mu(n_0^x - n_0^A) \gamma_x = \text{flux of nutrient } x \text{ assimilated}\)

Note: \(P_1^* + P_2^* = (j_A^* + j_B^*)/\mu \text{ (mass conservation)}\)

specialist
vs specialist

generalist
vs generalist
let $\psi_i = \frac{f_i^*}{(p_i + p_i^*)}$; frac. abundance of sp. i

$f_x^* = \frac{d^x}{(j_x^* + j_x^*)}$; frac. assim. flux for nutrient $x$

then $\psi_i = f_x^* \frac{V_{2B}}{V_{2B} - V_{1B}} - f_B \frac{V_{2A}}{V_{1A} - V_{2A}}$ from mass conservation

In limit $\mu \to 0$ (emphasizes effect of competition)

$\eta^*_x \propto \mu \ll \eta^*_x$

then $\hat{j}_x = \mu(\eta^*_x - \eta^*_x) Y_x \propto \mu \eta^*_x Y_x \propto$ environmental parameter.

$\psi_i = f_A \frac{V_{2B}}{V_{2B} - V_{1B}} - f_B \frac{V_{2A}}{V_{1A} - V_{2A}}$

where $f_x = \frac{\hat{j}_x}{\eta^*_x} = \frac{\eta^*_x Y_x}{\eta^*_x Y_A + \eta^*_x Y_B}$ env. parameters

let $M_x = \frac{\eta^*_x}{\eta^*_x}$ be uptake preference of species 1 for nutrient $x$ rel. to species 2 for $x$

$\to \psi_i = \frac{f_A}{1 - m_B} - \frac{f_B}{m_A - 1}$

- Condition for Coexistence: $1 > \psi_i > 0$
  - if $M_A > 1$, $M_B > 1$, $\psi_i < 0$ \{no coexistence\}
  - if $M_A < 1$, $M_B < 1$, $\psi_i > 1$ \{\}
  - for $M_A > 1 > M_B$ or $M_A < 1 < M_B$

$\psi_i > 0 \quad \frac{f_A}{1 - m_B} > \frac{f_B}{m_A - 1}$,

$f_A M_A - f_A > f_B - m_B f_B \to f_A M_A + f_B m_B > 1$
\[ \psi_i < 1. \frac{f_A - f_B}{1 - m_B} < 1 \]
\[ m_A m_B < m_A m_B + m_B f_A \Rightarrow \frac{f_A}{m_B} + \frac{f_B}{m_B} > 1 \]

\[ \Rightarrow \text{Ecological phase diagram: range of } f_A \text{ given } (m_A, m_B) \]

\[ m_A > 1 > m_B \text{ or } m_B > 1 > m_A \]
\[ f_A m_B (1 - f_a) m_B > 1 \]
\[ \text{if } m_A > m_B \text{ then } f_A > \frac{1 - m_B}{m_A - m_B} \]
\[ \text{if } m_A < m_B \text{, then } f_A < \frac{m_B}{m_B - m_A} \]
\[ f_A m_B (1 - f_a) m_B > 1 \]
\[ \text{if } m_B > m_A \text{, then } f_A > \frac{m_A (m_B - 1)}{m_B - m_A} \]
\[ \text{if } m_A > m_B \text{, then } f_A < \frac{m_A (1 - m_B)}{m_A - m_B} \]

\[ \text{for } m_B < 1 < m_A: \]
\[ \frac{1 - m_B}{m_A - m_B} < f_A < \frac{m_A}{m_A - m_B} \]
\[ (\psi_i) = (0) \text{ coexistence } (\psi_i) = (0) \]
\[ 0 \quad \frac{1 - m_B}{m_A - m_B} \quad \frac{m_A}{m_A - m_B} \quad 1 \]

\[ \Rightarrow \text{Coexistence occurs for intermediate range of } f_A \text{ but meaning of condition obscure (see later)} \]
(b) phenotypical phase diagram for coexistence for fixed $f_a$

$\Psi > 0$: $m_a f_a + m_b f_b > 1$

$\Psi < 1$: $\frac{f_a}{m_a} + \frac{f_b}{m_b} > 1$

- Conditions favorable for coexistence:
  - Large $m_a$, small $m_b$ or vice versa
  - = niche specialization

- For a given $f_a$, species can change genetic parameter ($m_a, m_b$) to drive the other species to extinction! Thus, phenotypical phase diagram has element of "fitness landscape" (but $\Psi$ ≠ fitness)

→ challenge: $f_a$ is variable

- however, trivial effect for overall scale of $V_{i|x}$:
  - If $V_{i|x} > V_{i|a}$, $f_x \to 0$
    
  - If $V_{i|x} < V_{i|a}$, $f_i \to 0$

→ need to "fix" the scale of $V_{i|x}$